Complex hyperbolic triangle groups

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Abstract

We prove several discreteness and non-discreteness results about complex hyperbolic triangle groups and discover two new lattices. These results use geometric (explicit construction of a fundamental domain), group theoretic and arithmetic methods.

Declaration

The work in this thesis is based on research carried out at the Department of Mathematical Sciences, England. No part of this thesis has been submitted elsewhere for any other degree or qualification and it all my own work unless referenced to the contrary in the text.

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Chapter 1

Introduction

1.1 Summary of Results

There are three main topics in this thesis, they make up chapters 2, 3 and 4. Chapter 1 contains background material and chapter 5 collects several miscellaneous results and various ideas for future work.

In chapter 2 we examine deformed triangle groups and prove several discreteness and non-discreteness results. We first describe a method for identifying deformed triangle groups; it turns out that, after relabelling generators, many of these groups are the same. Then we prove results based on a version of Jørgensen's inequality which allows us to immediately show that many deformed triangle groups are non-discrete. Then we describe a computer programme that preforms a brute force search using these non-discreteness tests, this yields a number of interesting groups for which we then determine whether or not they are discrete. In particular, we discover two new deformed triangle groups that are also lattices, in the language of this thesis they are $\Gamma(3, 3, 4; 7)$ and $\Gamma(3, 3, 5; 5)$.

In chapter 3 we prove that these groups are lattices by showing that they are commensurable to Mostow-Deligne lattices. Using the information from the coset decomposition, we produce presentations for these groups and calculate their co-volume. Finally we describe a fundamental domain for the groups which has the same covolume and gives the same presentation for the group after using Poincaré's polyhedron theorem. We do not go through all the necessary steps to ensure that these fundamental domains are geometrically correct, however in the next chapter we do this for another group and same methods could in principle be used for these groups.

In chapter 4 we analyse Deraux's lattice [6] ($\Gamma(4,4,4;5)$ in our notation). In particular we construct a fundamental domain for the group and perform all the necessary checks and calculations to ensure that the domain satisfies the hypotheses of Poincaré's polyhedron theorem. Then we use the theorem to produce a presentation and calculate the covolume of the group.

In chapter 5 we collect some unrelated results, namely a list of, to the best of our knowledge, all known discrete deformed triangle groups and a possible approach for groups generated by higher order reflections.

1.2 Preliminaries

Most of the material in this section is completely standard, see [10] for additional details. Consider $\mathbb{C}^{n,1}$, a copy of $\mathbb{C}^{n+1}$ equipped with an Hermitian form of signature $(n,1)$. Complex hyperbolic n-space, denoted $\mathbf{H}^n_{\mathbb{C}}$, is the complex projectivisation of the negative vectors in $\mathbb{C}^{n,1}$ with respect to the Hermitian product. Explicitly, let H be an Hermitian matrix with signature $(n,1)$ and define $\langle \mathbf{v}, \mathbf{w} \rangle = \mathbf{w}^* H \mathbf{v}$ to be the **Hermitian product**. We call a vector $\mathbf{v}$, **positive**, (respectively **null**, **negative**) if its Hermitian norm, $\langle \mathbf{v}, \mathbf{v} \rangle$, is positive (respectively zero, negative). Then $\mathbf{H}^n_{\mathbb{C}}$ is the the projectivisation of the set of negative vectors. The boundary of $\mathbf{H}^n_{\mathbb{C}}$, denoted $\partial \mathbf{H}^n_{\mathbb{C}}$, is the projectivisation of the set of null vectors. Throughout this thesis, we assume $n = 2$.

Definition 1.2.1 *A general point $z \in \overline{\boldsymbol{H}^2_{\mathbb{C}}} = \boldsymbol{H}^2_{\mathbb{C}} \cup \partial \boldsymbol{H}^2_{\mathbb{C}}$ is an ordered pair of complex numbers (z_1, z_2). We say the* ***standard lift*** *of z is the vector $\boldsymbol{z} = (z_1, z_2, 1)^t$. If z and w are two points, then we can define their Hermitian product as $\langle \boldsymbol{z}, \boldsymbol{w} \rangle = \boldsymbol{w}^* H \boldsymbol{z}$, where $*$ denotes complex transpose. Since we are only interested in $\mathbb{C}^{n,1}$ up to projectivisation, we may multiply the standard lift of a point by a non-zero complex number without changing the corresponding point in $\boldsymbol{H}^2_{\mathbb{C}}$. If $\boldsymbol{z}$ is the standard lift of a point z in $\boldsymbol{H}^2_{\mathbb{C}}$, then we say $\lambda \boldsymbol{z}$ is a generic* ***lift*** *of z, for $\lambda \in \mathbb{C} \setminus \{0\}$.*

There are two common choices for the Hermitian form,

$$H = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \widehat{H} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{1.1}$$

It is a straightforward calculation to show that when we choose H as our Hermitian form, the standard lift of a point in complex hyperbolic plane $\mathbf{H}^2_{\mathbb{C}}$ is of the form $(z_1, z_2, 1)^t$ where $|z_1|^2+|z_2|^2< 1$. In other words $\mathbf{H}^2_{\mathbb{C}}$ is the unit 4-ball in $\mathbb{C}^2$ and $\partial\mathbf{H}^2_{\mathbb{C}}$ is its 3-sphere boundary.

If we had started with $\mathbb{C}^{1,1}$ and taken $H = \mathrm{diag}(1, -1)$, this construction would lead to the Poincaré disc model of the real hyperbolic plane. This is a consequence of the isometry between $\mathbf{PU}(1, 1)$ and $\mathbf{SL}_2(\mathbb{R})$, from which it follows that $\mathbf{H}^1_{\mathbb{C}} = \mathbf{H}^2_{\mathbb{R}}$. We call this model of $\mathbf{H}^2_{\mathbb{C}}$ the **ball model**.

When we chose the second Hermitian form, $\widehat{H}$, the standard lift of a point in $\mathbf{H}^2_{\mathbb{C}}$ satisfies $2\mathrm{Re}\,(z_1) + |z_2|^2< 0$. The two dimensional analogue of this construction leads to the Poincaré upper half model of the real hyperbolic plane. We call this model of $\mathbf{H}^2_{\mathbb{C}}$ the **paraboloid model**.

Definition 1.2.2 *The metric on $\boldsymbol{H}^2_{\mathbb{C}}$ is the **Bergman metric**, it depends only on the Hermitian form and can be expressed explicitly as follows,*

$$\cosh^2\left(\frac{d(z, w)}{2}\right) = \frac{\langle \boldsymbol{z}, \boldsymbol{w}\rangle\langle \boldsymbol{w}, \boldsymbol{z}\rangle}{\langle \boldsymbol{z}, \boldsymbol{z}\rangle\langle \boldsymbol{w}, \boldsymbol{w}\rangle}.$$

The group of holomorphic isometries of $\mathbf{H}^2_{\mathbb{C}}$ is $\mathbf{PU}(2, 1)$, that is, unitary matrices that preserve a signature $(2, 1)$ Hermitian form H. As with real hyperbolic geometry, we can partition these isometries into the standard triptych. **Elliptic** isometries have at least one fixed point in $\mathbf{H}^2_{\mathbb{C}}$, **parabolic** isometries have exactly on fixed point which lies in in the boundary $\partial\mathbf{H}^2_{\mathbb{C}}$ and **loxodromic** isometries have exactly two fixed points both of which lie $\partial\mathbf{H}^2_{\mathbb{C}}$.

Definition 1.2.3 *Let z and w be two points in $\overline{\boldsymbol{H}^2_{\mathbb{C}}}$, we denote the geodesic containing these two points, with respect to the Bergmann metric defined above, as $\gamma(z, w)$. The geodesic segment with endpoints z and w we denote $[z, w]$.*

Since $\boldsymbol{H}^2_{\mathbb{C}}$ is a Riemannian manifold with everywhere negative curvature $\gamma(z,w)$ is unique. By taking $H = \mathrm{diag}(1,1,-1)$, we normalise so that the sectional curvatures, κ lie in the interval, $1 \leq \kappa \leq -1/4$.

Proposition 1.2.4 *Let z and w be two points in $\boldsymbol{H}^2_{\mathbb{C}}$, we can rescale their standard lifts to get, $\boldsymbol{z}$ and $\boldsymbol{w}$, where $\langle \boldsymbol{z},\boldsymbol{z}\rangle = \langle \boldsymbol{w},\boldsymbol{w}\rangle = -1$ and $\langle \boldsymbol{z},\boldsymbol{w}\rangle \in \mathbb{R}_{<0}$. Then the geodesic between z and w is given by the projectivisation of the vectors*

$$\gamma(z,w) = \{\sinh((t-s)/2)\boldsymbol{z} + \sinh((r-t)/2)\boldsymbol{w} \mid t \in \mathbb{R}\}, \tag{1.2}$$

where s and r are two distinct real numbers such that $|s-r| = d(z,w)$ and t is the variable parametrising the geodesic.

Definition 1.2.5 *Let z and w be two points in $\overline{\boldsymbol{H}^2_{\mathbb{C}}}$, we define $\mathcal{C}(z,w)$ to be the* ***complex geodesic*** *or $\mathbb{C}$-line containing these two points. Let $\boldsymbol{z}$ and $\boldsymbol{w}$ be the lifts of z and w. Take $\boldsymbol{C}$ to be the complex span of $\boldsymbol{z}$ and $\boldsymbol{w}$ in the vector space $\mathbb{C}^{2,1}$. Then we define $\mathcal{C}(z,w)$ be the complex projectivisation of $\boldsymbol{C}$. It is clear that $\mathcal{C}(z,w)$ is a 2 dimensional subspace of $\boldsymbol{H}^2_{\mathbb{C}}$ and that it is well defined. Furthermore, any complex line $\mathcal{C}$ is the image of $\mathcal{C}((1,0),(0,0)) = \{(z_1,0) \mid z_1 \in \mathbb{C},\ |z_1| < 1\}$ under some element of $\mathbf{PU}(2,1)$.*

Lemma 1.2.6 *When we restrict the Bergmann metric to any $\mathbb{C}$-line, $\mathcal{C}$, we see that $\mathcal{C}$ is a simply connected, 2-dimensional manifold with constant Riemannian curvature $\kappa = -1$. In other words a $\mathbb{C}$-line is a copy of $\boldsymbol{H}^2_{\mathbb{R}}$ embedded in $\boldsymbol{H}^2_{\mathbb{C}}$. When $H = \mathrm{diag}(1,1,-1)$ the restricted metric is exactly the usual Poincaré disc metric. Given two points z and w, the geodesic $\gamma(z,w)$ is a one real dimensional subspace of $\mathcal{C}(z,w)$, i.e. $\mathbb{C}$-lines are totally geodesic.*

Definition 1.2.7 *The* ***polar vector*** *to a $\mathbb{C}$-line $\mathcal{C}$ is the unique vector (up to scalar multiplication) perpendicular to $\boldsymbol{C}$ in $\mathbb{C}^{2,1}$ with respect to the Hermitian form. A polar vector is always a positive vector and every positive vector corresponds to a complex geodesic.*

Definition 1.2.8 *Let γ be a geodesic contained in a $\mathbb{C}$-line $\mathcal{C}$. A* ***hypercycle****, δ, is a curve in $\mathcal{C}$ such that the orthogonal distance between every point on δ and the geodesic γ is constant.*

Definition 1.2.9 *Given a $\mathbb{C}$-line $\mathcal{C}$ with polar vector $\boldsymbol{n}$, there is a unique involution $I \in \mathbf{PU}(2,1)$, that fixes every point in $\mathcal{C}$. We call I the* ***complex involution*** *(or order 2 complex reflection) in $\mathcal{C}$. Explicitly I is given by*

$$I(\boldsymbol{z}) = -\boldsymbol{z} + 2\frac{\langle \boldsymbol{z}, \boldsymbol{n}\rangle}{\langle \boldsymbol{n}, \boldsymbol{n}\rangle}\boldsymbol{n}. \tag{1.3}$$

Definition 1.2.10 *There are also higher order reflections in $\mathcal{C}$. Let ζ be a complex number of absolute value one. Consider the isometry,*

$$I(\boldsymbol{z}, \zeta) = -\boldsymbol{z} + (1-\zeta)\frac{\langle \boldsymbol{z}, \boldsymbol{n}\rangle}{\langle \boldsymbol{n}, \boldsymbol{n}\rangle}\boldsymbol{n}. \tag{1.4}$$

This map fixes $\mathcal{C}$ and rotates the orthogonal direction by θ where $\zeta = e^{i\theta}$. When θ is a rational multiple of π, the reflection I will have finite order. If we set $\zeta = -1$, we recover the formula for the order two complex reflection (1.3).

Definition 1.2.11 *As with the isometries of the Poincaré disc, we can divide the isometries of $\boldsymbol{H}^2_{\mathbb{C}}$ into three catagories depending on their fixed point sets. We say an isometry g is* ***loxodromic*** *if it fixes a unique pair of points on $\partial\boldsymbol{H}^2_{\mathbb{C}}$,* ***parabolic*** *if has a unique fixed point on $\partial\boldsymbol{H}^2_{\mathbb{C}}$ and* ***elliptic*** *if it has a fixed point in $\boldsymbol{H}^2_{\mathbb{C}}$. Furthermore we say an elliptic isometry is* ***regular elliptic*** *if and only if all its eigenvalues are distinct. If a parabolic isometry is not unipotent, then there is a unique $\mathbb{C}$-line on which the isometry acts as a parabolic isometry of $\boldsymbol{H}^1_{\mathbb{C}}$, we call such isometries* ***ellipto-parabolic****.*

Theorem 1.2.12 (Goldman trace formula) *(Theorem 6.2.4 of [10])*
Let $\omega = (-1+\sqrt{-3})/2$, define $f : \mathbb{C} \to \mathbb{R}$ as

$$f(z) = |z|^4 - 8\mathrm{Re}\,(z^3) + 18|z|^2 - 27. \tag{1.5}$$

Let A be a matrix in $\mathbf{SU}(2,1)$ with trace τ. Then,

1. *A is regular elliptic if and only if $f(\tau) < 0$;*

2. *A is loxodromic if and only if $f(\tau) > 0$;*

3. *A is ellipto-parabolic if and only if A is not elliptic (i.e. no fixed point in $\boldsymbol{H}^2_{\mathbb{C}}$), $f(\tau) = 0$ and $\tau \notin \{3, 3\omega, 3\overline{\omega}\}$;*

4. *A is a complex reflection, about either a point or a complex geodesic, if and only if A is elliptic,* $f(\tau) = 0$ *and* $\tau \notin \{3, 3\omega, 3\overline{\omega}\}$.

5. *A is a nilpotent automorphism of* $\boldsymbol{H}^2_{\mathbb{C}}$ *if and only if* $\tau \in \{3, 3\omega, 3\overline{\omega}\}$

Definition 1.2.13 *In addition to points, geodesics and* $\mathbb{C}$*-lines there is one more type of totally geodesic submanifold,* ***Lagrangians*** *or* $\mathbb{R}$*-planes. Lagrangians are totally real subspaces. Let* $\mathcal{R}$ *be a Lagrangian, then for all* z, $w \in \mathcal{R}$, *the Hermitian product of their standard lifts is real i.e.* $\langle \boldsymbol{z}, \boldsymbol{w} \rangle \in \mathbb{R}$. *Any Lagrangian* $\mathcal{R}$ *is the image of* $\{(z_1, z_2) \mid z_1, z_2 \in \mathbb{R}\}$ *under some element of* $\mathbf{PU}(2,1)$. *In the ball model the Bergmann metric restricted to any Lagrangian* $\mathcal{R}$ *is, up to a scalar multiple, the Klein-Beltrami metric for the unit disc model of* $\boldsymbol{H}^2_{\mathbb{R}}$.

Proposition 1.2.14 *These four classes of subspaces (points, geodesics,* $\mathbb{C}$*-lines and* $\mathbb{R}$*-planes) are the only totally geodesic subspaces of* $\boldsymbol{H}^2_{\mathbb{C}}$. *In particular there are no 3 dimensional totally geodesic hypersurfaces in* $\boldsymbol{H}^2_{\mathbb{C}}$.

Proposition 1.2.15 *Two distinct* $\mathbb{C}$*-lines with non empty intersection, intersect in exactly one point.*
Two distinct Lagrangians with non empty intersection, intersect in a point or a geodesic.
A Lagrangian and a $\mathbb{C}$*-line with non empty intersection, intersect in a point or a geodesic.*

We now prove a few technical lemmas about the orthogonal projection onto a $\mathbb{C}$-line that we shall use in later chapters.

Lemma 1.2.16 (page 101 of [10]) *Let* $\mathcal{C}_1$ *and* $\mathcal{C}_2$ *be two* $\mathbb{C}$*-lines in* $\boldsymbol{H}^2_{\mathbb{C}}$ *and* $\Pi_1 : \boldsymbol{H}^2_{\mathbb{C}} \to \mathcal{C}_1$ *be the orthogonal projection map. Then there are three cases*

1. $\mathcal{C}_1$ *and* $\mathcal{C}_2$ *intersect in a point* x *contained in the interior of* $\boldsymbol{H}^2_{\mathbb{C}}$; *then* Π_1 *maps* $\mathcal{C}_2$ *diffeomorphically onto the geometric ball with centre* $x = \mathcal{C}_1 \cap \mathcal{C}_2$ *and radius* $2\tanh^{-1}\cos(\angle(\mathcal{C}_1, \mathcal{C}_2))$.

2. $\mathcal{C}_1$ *and* $\mathcal{C}_2$ *intersect in a point* x *contained in* $\partial\boldsymbol{H}^2_{\mathbb{C}}$; *then* Π_1 *maps* $\mathcal{C}_2$ *diffeomorphically onto a horoball centred at* $x = \overline{\mathcal{C}_1} \cap \overline{\mathcal{C}_2}$.

3. *$\mathcal{C}_1$ and $\mathcal{C}_2$ do not intersect in $\overline{\boldsymbol{H}^2_{\mathbb{C}}}$; let x be the unique closest point on $\mathcal{C}_1$ to $\mathcal{C}_2$, then Π_1 maps $\mathcal{C}_2$ diffeomorphically onto the geodesic ball with centre x and radius*

$$2\tanh^{-1}\operatorname{sech}\left(\frac{d(\mathcal{C}_1,\mathcal{C}_2)}{2}\right).$$

Lemma 1.2.17 *Assume that we are using the ball model of $\boldsymbol{H}^2_{\mathbb{C}}$ (i.e. $H = \operatorname{diag}(1,1,-1)$). Let γ be a geodesic, $\mathcal{C}$ a $\mathbb{C}$-line, and $\Pi_{\mathcal{C}} : \boldsymbol{H}^2_{\mathbb{C}} \to \mathcal{C}$ the orthogonal projection map onto $\mathcal{C}$. Then $\Pi_{\mathcal{C}}(\gamma)$ is the arc of a geometric circle in $\mathcal{C}$ or a single point.*

Proof: We may assume that $\mathcal{C} = \{(\zeta,0,1)^t | \zeta \in \mathbb{C}\}$ and the geodesic γ passes through the points p and q with standard lifts $\mathbf{p} = (0,k,1)^t$ and $\mathbf{q} = (z_1,z_2,1)^t$ with $k \in \mathbb{R}_{\geq 0}$ and $z_1, z_2 \in \mathbb{C}$. We also assume $z_1 \neq 0$ otherwise the geodesic will be contained in the $\mathbb{C}$-line $(0,\zeta,1)^t$, so γ will be projected to a single point. We need to rescale $\mathbf{q}$ by a factor of $\lambda = 2(1-k\overline{z_2})/|1-k\overline{z_2}|^2$ in order to use (1.2). After doing this the geodesic between these two points is

$$\gamma(p,q) = \{\sinh(t-s)\mathbf{p} + \sinh(r-t)\lambda\mathbf{q} \mid t \in \mathbb{R}\}.$$

Recall, s and r are constants dependent on p, q that ensure the curve is parametrised correctly and t is the parameter.

When we orthogonally project onto $\mathcal{C}$, we throw away the second coordinate in the vector and leave the other entries unchanged.

$$\Pi_{\mathcal{C}}(\gamma(p,q)) = \begin{pmatrix} \sinh(r-t)\lambda z_1 \\ 0 \\ \sinh(t-s)+\sinh(r-t)\lambda \end{pmatrix} \sim \begin{pmatrix} \dfrac{\sinh(r-t)\lambda x}{\sinh(t-s)+\sinh(r-t)\lambda} \\ 0 \\ 1 \end{pmatrix}.$$

The term in the first entry of the normalised vector defines a circle. To see this let $\sinh(t-s) = A$ and $\sinh(r-t)\lambda = B$, then the expression we need to understand is $Bx/(A+B)$. We invert this expression in the unit circle in the complex plane, i.e. we apply the map $f : z \mapsto \overline{z}^{-1}$.

$$\begin{aligned} \overline{\left(\frac{Bz_1}{A+B}\right)}^{-1} &= \frac{1}{\overline{z_1}}\left(1+\frac{\overline{A}}{\overline{B}}\right) \\ &= \frac{1}{\overline{z_1}}\left(1+\frac{\sinh(t-s)}{\sinh(r-t)}\frac{1}{\overline{\lambda}}\right). \end{aligned} \tag{1.6}$$

This is the equation of a straight line in the complex plane. The inversion of $Bz_1/(A+B)$ is a straight line, so $Bz_1/(A+B)$ is a circle passing through the origin (or a straight line through the origin). We can explicitly write down the equation of the circle,

$$S_0 = \left\{ \frac{-iz_1}{2k} \frac{(1-\overline{z_2}k)}{\operatorname{Im}(z_2)} (1+e^{i\theta}) \mid \theta \in [0, 2\pi) \right\},$$

then

$$\Pi_{\mathcal{C}}(\gamma(p,q)) = \begin{pmatrix} S_0 \\ 0 \\ 1 \end{pmatrix}.$$

To see this is true for general points p, q and a general $\mathbb{C}$-line $\mathcal{C}$, recall that a $\mathbb{C}$-line is an embedded copy of $\mathbf{H}^2_{\mathbb{R}} = \mathbf{H}^1_{\mathbb{C}}$, so an isometry of $\mathbf{H}^2_{\mathbb{C}}$ that preserves a $\mathbb{C}$-line, acts isometrically on that $\mathbb{C}$-line. Isometries of $\mathbf{H}^2_{\mathbb{R}}$ are Möbius transformations, in particular they send circles to circles.

When $k = 0$ or $\operatorname{Im}(z_2) = 0$ the circle is a straight line. In the former case, this means that $\gamma(p,q) \cap \mathcal{C} \neq \emptyset$, in the latter p, q, $\Pi_{\mathcal{C}}(p)$ and $\Pi_{\mathcal{C}}(q)$ are all contained in an $\mathbb{R}$-plane. □

Corollary 1.2.18 *Let $\mathcal{C}$ be a $\mathbb{C}$-line, γ be a geodesic and $\Pi_{\mathcal{C}}$ be the orthogonal projection onto $\mathcal{C}$. If $\gamma \cap \mathcal{C} \neq \emptyset$, then $\Pi_{\mathcal{C}}(\gamma)$ is a geodesic in $\mathcal{C}$.*

PROOF: The image of the geodesic under $\Pi_{\mathcal{C}}$ will be a straight line segment in $\mathcal{C}$ that passes through the origin. Since $\mathcal{C}$ is a copy of the Poincaré disc, it follows that $\Pi_{\mathcal{C}}(\gamma)$ is a geodesic in $\mathcal{C}$. □

Lemma 1.2.19 *Using the normalisation from lemma 1.2.17, i.e. a $\mathbb{C}$-line $\mathcal{C} = \{(\zeta, 0, 1)^t | \zeta \in \mathbb{C}\}$, points p and q with standard lifts $\boldsymbol{p} = (0, k, 1)^t$ and $\boldsymbol{q} = (z_1, z_2, 1)^t$ with $k \in \mathbb{R}_{\geq 0}$ and $z_1, z_2 \in \mathbb{C}$, the geodesic segment $[p, q]$ projects to the shorter arc of S_0 (as defined in lemma 1.2.17) between $\Pi_{\mathcal{C}}(p) = 0$ and $\Pi_{\mathcal{C}}(q) = z_1$.*

PROOF: To see that $\Pi_{\mathcal{C}}(\gamma)$ projects onto the shortest arc of S_0 we make the following observation: $\sinh(t-s)/\sinh(r-t)$ is always positive for $t \in (r, s)$, furthermore

$\sinh(t-s)/\sinh(r-t)$ is a bijective function $(r,s) \to (0,\infty)$. Let $t' = \sinh(t-s)/\sinh(r-t) > 0$, then the line (1.6) becomes $(1+t'/\overline{\lambda})/\overline{z_1}$ for $t' > 0$. The image of this half line under the reciprocal conjugate map is the relevant arc of S_0.

When $t' = -1$ (respectively $t' = 0$, $t' = \infty$) the corresponding point on the reciprocal conjugate of the line is the point diametrically opposite the origin $(-iz_1(1-\overline{z_2}k))/(2k\mathrm{Im}\,(z_2))$, (respectively z_1, 0). So the relevant circle arc is contained entirely in half the circle and so must be the shortest arc. □

Lemma 1.2.20 *Let z_1, z_2 and k be as in lemma 1.2.17, then,*

$$\left|\frac{(1-\overline{z_2}k)}{\mathrm{Im}\,(z_2)}\right| \geq 1. \tag{1.7}$$

More usefully,

$$\left|\frac{-iz_1}{2k}\frac{(1-\overline{z_2}k)}{\mathrm{Im}\,(z_2)}\right| \geq \frac{|z_1|}{2k}. \tag{1.8}$$

Proof: Let $z_2 = a + ib$ then after squaring (1.7) we get

$$\begin{aligned}\frac{|1-k(a-ib)|^2}{b^2} &= \frac{(1-ka)^2 + k^2b^2}{b^2}\\ &= \frac{(1-ka)^2}{b^2} + k^2.\end{aligned}$$

Since $|z_2|^2 \leq 1$, $b^2 \leq 1 - a^2$, so.

$$\frac{|1-k(a-ib)|^2}{b^2} \geq \frac{(1-ka)^2}{1-a^2} + k^2$$

After differentiating with respect to a we find that the expression on the right attains its minimum on the interval $[0,1]$ at $a = k$, so

$$\begin{aligned}\frac{|1-k(a-ib)|^2}{b^2} &\geq \frac{(1-k^2)^2}{1-k^2} + k^2\\ &= 1.\end{aligned}$$

□

In some sense this lemma measures how close the image of a projected geodesic onto a $\mathbb{C}$-line is to a geodesic. If k is very small (p is very close to $\mathcal{C}$) then the radius of the circle passing through the origin will have to be very large, and the projected geodesic will be very close to a straight line through the origin, i.e. a geodesic.

Corollary 1.2.21 *In the ball model of $\boldsymbol{H}^2_{\mathbb{C}}$, let $\mathcal{C}_1$ and $\mathcal{C}_2$ be $\mathbb{C}$-lines, $\delta \subset \mathcal{C}_2$ a hypercycle in $\mathcal{C}_2$ and $\Pi_1 : \boldsymbol{H}^2_{\mathbb{C}} \to \mathcal{C}_1$ be the orthogonal projection map onto $\mathcal{C}_1$. Then $\Pi_1(\delta)$ is an arc of a geometric circle in $\mathcal{C}_1$, or a point if $\mathcal{C}_1$ and $\mathcal{C}_2$ are orthogonal.*

PROOF: Assume without loss of generality that $\mathcal{C}_1$ is $\{(z,0,1)^t | z \in \mathbb{C}\}$ and $\mathcal{C}_2$ is a $\mathbb{C}$-line with polar vector $\mathbf{n} = (X, Y, 1)^t$. Assume $Y \neq 0$ then we can parametrise $\mathcal{C}_2$ as

$$\mathcal{C}_2 = \left\{ \left(\begin{array}{c} \zeta \\ \dfrac{1 - \zeta\overline{X}}{\overline{Y}} \\ 1 \end{array} \right) \mid \zeta \in \mathbb{C} \right\}.$$

In the Poincaré disc model of $\mathbf{H}^2_{\mathbb{R}}$ a hypercycle is the arc of a geometric circle that intersects the boundary, so any hypercycle in $\mathcal{C}$ an arc of

$$\delta = \left\{ \left(\begin{array}{c} r_0 e^{i\theta} + z_0 \\ \dfrac{1 - (r_0 e^{i\theta} + z_0)\overline{X}}{\overline{Y}} \\ 1 \end{array} \right) \mid \theta \in [0, 2\pi) \right\}.$$

for some $r_0 \in \mathbb{R}_{>0}$ and $z_0 \in \mathbb{C}$. The image under orthogonal projection onto $\mathcal{C}_1$ of δ is $(r_0 e^{i\theta} + z_0, 0, 1)^t$ which is clearly a geometric circle in $\mathcal{C}_1$. □

Lemma 1.2.22 *Consider the Poincaré disc model of $\boldsymbol{H}^2_{\mathbb{R}}$ as a subset of the complex plane. Let γ be a geodesic that does not pass through the origin. We normalise so that the end points of γ are $e^{i\theta}$ and $-e^{-i\theta}$. Let z_0 a point in the disc other than the origin and S be a circle that contains both z_0 and the origin. Let S' be shortest arc of the S between z_0 and the origin. Let*

$$T_\theta := \{z \in \mathbb{C} \mid \cos(\theta)^2 = (\mathrm{Im}\,(z)\sin(\theta) - 1)^2 - \mathrm{Re}\,(z)^2 \cos^2(\theta)\}. \tag{1.9}$$

The curve T divides $\boldsymbol{H}^2_{\mathbb{R}}$ into two regions. If z_0 lies in the same half as the origin, then $S' \cap \gamma = \emptyset$.

PROOF: For the sake of clarity we use the coordinate system $(x, y) = (\mathrm{Re}\,(z), \mathrm{Im}\,(z))$ for $\mathbb{C}$. Then the Poincaré disc is $\{(x, y) \mid x^2 + y^2 < 1\}$ and the geodesic γ is part of the circle defined by the equation $x^2 + (y - \sin(\theta))^2 = (1 + \cos(\theta))^2/\sin^2(\theta)$, the radius of this circle is $\cos(\theta)/\sin(\theta)$.

Let (x_0, y_0) be the point z_0 and $\widehat{S}$ be the circle whose diameter is the line between (x_0, y_0) and $(0, 0)$. The centre of $\widehat{S}$ is $(x_0/2, y_0/2)$. Let $L = \sqrt{(x_0/2)^2 + (y_0/2)^2}$, $M = \cos(\theta)/\sin(\theta)$ and $N = \sqrt{(x_0/2)^2 + ((y_0/2) - 1/\sin(\theta))^2}$, the circle $\widehat{S}$ is tangent to γ when $L + M = N$. When $L + M < N$ (respectively $L + M > N$), $\widehat{S}$ does not intersect γ (respectively intersects γ twice), see figure 1.1.

$$\begin{aligned}
L + M &= N \\
\sqrt{\left(\frac{x_0}{2}\right)^2 + \left(\frac{y_0}{2}\right)^2} + \frac{\cos(\theta)}{\sin(\theta)} &= \sqrt{\left(\frac{x_0}{2}\right)^2 + \left(\left(\frac{y_0}{2}\right) - \frac{1}{\sin(\theta)}\right)^2} \\
\left(\frac{y_0}{2}\right)^2 + 2\frac{\cos(\theta)}{\sin(\theta)}\sqrt{\left(\frac{x_0}{2}\right)^2 + \left(\frac{y_0}{2}\right)^2} + \frac{\cos^2(\theta)}{\sin^2(\theta)} &= \left(\left(\frac{y_0}{2}\right) - \frac{1}{\sin(\theta)}\right)^2 \\
2\frac{\cos(\theta)}{\sin(\theta)}\sqrt{\left(\frac{x_0}{2}\right)^2 + \left(\frac{y_0}{2}\right)^2} + \frac{\cos^2(\theta)}{\sin^2(\theta)} &= -\frac{y_0}{\sin(\theta)} + \frac{1}{\sin^2(\theta)} \\
2\frac{\cos(\theta)}{\sin(\theta)}\sqrt{\left(\frac{x_0}{2}\right)^2 + \left(\frac{y_0}{2}\right)^2} &= 1 - \frac{y_0}{\sin(\theta)} \\
2\frac{\cos^2(\theta)}{\sin^2(\theta)}\left(\left(\frac{x_0}{2}\right)^2 + \left(\frac{y_0}{2}\right)^2\right) &= 1 - 2\frac{y_0}{\sin(\theta)} + \frac{y_0^2}{\sin^2(\theta)} \\
\cos^2(\theta)(x_0^2 + y_0^2) &= \sin^2(\theta) - 2y_0\sin(\theta) + y_0^2 \\
\cos^2(\theta) &= (y_0\sin(\theta) - 1)^2 - x_0^2\cos^2(\theta).
\end{aligned}$$

Substituting $\mathrm{Re}\,(z_0) = x_0$ and $\mathrm{Im}\,(z_0) = y_0$, gives the required result. $\square$

1.2.1 Bisectors

There are no totally geodesic submanifolds of $\mathbf{H}^2_{\mathbb{C}}$ other than the four subspaces mentioned in Proposition 1.2.14. The lack of totally geodesic 3-dimensional subspaces complicates matters when constructing fundamental domains, since we cannot construct the boundary of such a region from pieces of totally geodesic submanifolds. However there is a three dimensional submanifold that is foliated by totally geodesic subspaces in two different ways.

Definition 1.2.23 *Given two points z, $w \in \boldsymbol{H}^2_{\mathbb{C}}$, the* ***bisector*** *$\mathcal{B}(z, w)$ is defined as*

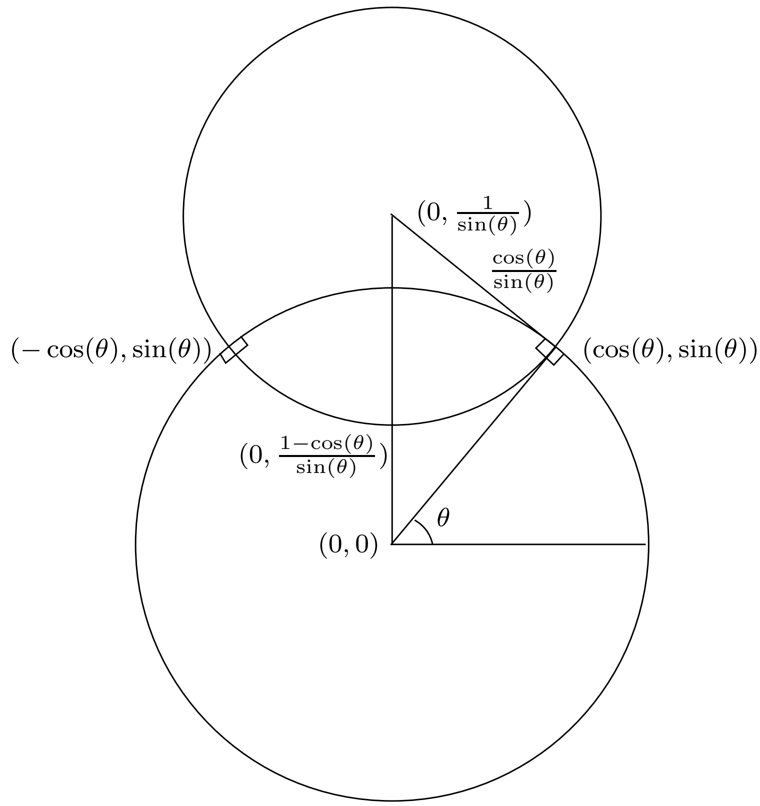

Figure 1.1: Schematic for the proof of lemma 1.2.22

the set of points equidistant from z and w,

$$\mathcal{B}(z,w) = \{x \in \boldsymbol{H}^2_{\mathbb{C}} \mid d(x,z) = d(x,w)\}.$$

The ***complex spine*** *of $\mathcal{B}(z,w)$, denoted Σ, is the complex geodesic $\mathcal{C}(z,w)$. The* ***real spine*** *of $\mathcal{B}(z,w)$, denoted σ, is the real geodesic contained in Σ that is equidistant from z and w. Alternatively $\sigma = \mathcal{B}(z,w) \cap \Sigma$.*

Proposition 1.2.24 *In $\boldsymbol{H}^2_{\mathbb{C}}$ a bisector is a smooth 3 dimensional hypersurface diffeomorphic to $\mathbb{R}^3$.*

Theorem 1.2.25 (Mostow, Goldman) *([20], [10])*

1. *Let $\Pi_\Sigma : \boldsymbol{H}^2_{\mathbb{C}} \to \Sigma$, be the orthogonal projection map onto Σ. Then $\mathcal{B}$ is foliated by complex geodesics of the form $\Pi_\Sigma^{-1}(z)$ for $z \in \sigma$. These complex geodesics are called* ***complex slices*** *of $\mathcal{B}$.*

2. *The bisector $\mathcal{B}$ is the union of all Lagrangians that contain σ. These Lagrangians are called **meridians** of $\mathcal{B}$.*

Corollary 1.2.26 • *A bisector is uniquely determined by its real spine.*

- *The bisector $\mathcal{B}$ is fixed under a complex involution in any of its slices.*
- *The bisector $\mathcal{B}$ is fixed under an anti-holomorphic involution in any of its meridians.*

Intersection of bisectors

The intersection of two or more bisectors can be very complicated, in general it is not connected or contained in a totally geodesic subspace. We now collect several results which allow us to understand bisector intersections. Many of these results are taken from [7] or [10].

Proposition 1.2.27 *(2.4 of [7]) Let $\mathcal{B}$ be a bisector and $\mathcal{C}$ be a $\mathbb{C}$-line such that $\mathcal{B} \cap \mathcal{C} \neq \emptyset$, then $\mathcal{C} \subset \mathcal{B}$ (in which case $\mathcal{C}$ is a slice of $\mathcal{B}$) or $\mathcal{C} \cap \mathcal{B}$ is a hypercycle δ in $\mathcal{C}$. In the ball model a hypercycle is an arc of a Euclidean circle intersecting the boundary.*

Proposition 1.2.28 *Let $\mathcal{B}$ be a bisector and γ a geodesic, then either γ is entirely contained in a slice or meridian of $\mathcal{B}$ or $\gamma \cap \mathcal{B}$ consists of at most two points. Moreover the number of intersection points between γ and $\mathcal{B}$ is equal to the number of intersection points between σ_B (the real spine of $\mathcal{B}$) with $\Pi_\Sigma(\gamma)$ (the image of γ under orthogonal projection onto the complex spine of $\mathcal{B}$).*

Proposition 1.2.29 *Let $\mathcal{B}$ be a bisector, with complex spine Σ and γ a geodesic. If $\gamma \cap \Sigma \neq \emptyset$ and γ is not contained in $\mathcal{B}$, then γ intersects $\mathcal{B}$ in at most one point.*

Definition 1.2.30 *We can categorise pairs of bisectors into the following classes. Let B_1 and B_2 be bisectors with complex (respectively real) spines Σ_1 and Σ_2 (respectively σ_1 and σ_2)*

- *If $\Sigma_1 = \Sigma_2$ we say B_1 and B_2 are **cospinal***

- *If Σ_1 and Σ_2 intersect outside the real spines, we say B_1 and B_2 are **coequidistant***

- *If B_1 and B_2 share a common meridian, we say B_1 and B_2 are **comeridanal***

- *If B_1 and B_2 share a common slice, we say B_1 and B_2 are **cotranchal***

Proposition 1.2.31 *The intersection of a pair of coequidistant bisectors is a smooth disc, in particular it is connected.*

Theorem 1.2.32 (Giraud's theorem) *Suppose that B_1 and B_2 are two bisectors with respective complex spines Σ_1 and Σ_2, such that*

- *Σ_1 and Σ_1 are distinct*

- *$\Sigma_1 \cap B_2$=$\Sigma_2 \cap B_1 = \emptyset$*

*Then their intersection is smooth disc, moreover there is at most one other bisector containing $B_1 \cap B_2$ other than B_1 and B_2. We call such an intersection of bisectors a **Giraud disc**.*

Proposition 1.2.33 *Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be a pair of coequidistant bisectors. Their intersection is a Giraud disc $\mathcal{G}$ and there exists a third bisector $\mathcal{B}_0$ containing $\mathcal{G}$. Let $p_0 = \Sigma_1 \cap \Sigma_2$, $p_1 = \Sigma_0 \cap \Sigma_1$ and $p_2 = \Sigma_0 \cap \Sigma_2$. Then*

$$\begin{aligned}
\mathcal{B}_1 &= \mathcal{E}(p_0, p_1) = \{x \in \boldsymbol{H}^2_{\mathbb{C}} \mid d(x, p_0) = d(x, p_1)\},\\
\mathcal{B}_2 &= \mathcal{E}(p_0, p_2) = \{x \in \boldsymbol{H}^2_{\mathbb{C}} \mid d(x, p_0) = d(x, p_2)\},\\
\mathcal{B}_0 &= \mathcal{E}(p_1, p_2) = \{x \in \boldsymbol{H}^2_{\mathbb{C}} \mid d(x, p_1) = d(x, p_2)\},
\end{aligned}$$

and

$$\mathcal{G} = \{x \in \boldsymbol{H}^2_{\mathbb{C}} \mid d(x, p_1) = d(x, p_2) = d(x, p_0)\}.$$

1.2.2 Arithmetic discreteness criterion

We state a useful theorem which allows us to quickly show that certain subgroups of $\mathbf{SU}(2,1)$ are arithmetic.

Theorem 1.2.34 (Arithmeticity criterion (12.2.6 of [5])) *Let E be an algebraic number field obtained by taking a totally imaginary quadratic extension E/F of a totally real number field F over $\mathbb{Q}$, i.e. E is an algebraic number field with totally real subfield F such that $[E : F] = 2$. By construction the extension E/F is Galois. If $\sigma : F \to \mathbb{R}$ and $\tau : E \to \mathbb{C}$ are compatible embeddings (i.e. $\tau(E) \cap \mathbb{R} = \sigma(F)$) then the non-trivial involution of the Galois group* $\mathrm{Gal}(E/F)$ *is simply the restriction of complex conjugation in $\mathbb{C}$ to E.*

We denote the ring of integers of E by $\mathcal{O}_E$. Let H be an Hermitian form all of whose entries lie in $\mathcal{O}_E$ and $\mathbf{SU}(H)$ *the special unitary matrix group with respect to this Hermitian form. When H has signature* $(2, 1)$, $\mathbf{SU}(H) = \mathbf{SU}(2, 1)$, *in particular* $\mathbf{SU}(H)$ *is non-compact. Consider the group* $\mathbf{SU}(H, \mathcal{O}_E) \subset \mathbf{SU}(H)$, *i.e the matrix subgroup of* $\mathbf{SU}(H)$ *where the entries of every matrix lie in the ring of integers of E.*

There are a finite number of embeddings $\sigma_i : F \to \mathbb{R}$. For each of these embeddings there is, up to complex conjugation, a unique compatible embedding $\tau_i : E \to \mathbb{C}$, from which we obtain a new Hermitian matrix ${}^{\tau_j}H$ by applying τ_j to the entries of H and the new associated group $\mathbf{SU}({}^{\tau_j}H)$ *by* ${}^{\tau_j}\mathbf{SU}(H)$.

The group $\mathbf{SU}(H, \mathcal{O}_E)$ *is an arithmetic lattice in* $\mathbf{SU}(H)$ *if and only if* ${}^{\tau_j}\mathbf{SU}(H)$ *is compact for all non-trivial τ_j. This is equivalent to ${}^{\tau_j}H$ being an Hermitian form of signature* $(3, 0)$ *or* $(0, 3)$.

More generally any subgroup $G \subset \mathbf{SU}(H, \mathcal{O}_E)$ *will be a subgroup of a lattice and therefore discrete.*

This formulation of the theorem was taken from chapter 5 of Ben McReynolds excellent notes [19]. We refer any readers to these notes for further details about the background and proof of this theorem and arithmetic lattices of $\mathbf{SU}(n, 1)$ in general.

Chapter 2

Deformed $\mathbb{R}$-Fuchsian triangle groups

2.1 Triangle groups

In this chapter we introduce the topic of this thesis, deformed $\mathbb{R}$-Fuchsian triangle groups. We then collect some known results about these groups and prove some new discreteness results. We perform a computer search which yields two new lattices, several interesting discrete deformed triangle groups and two infinite families where we believe every group is discrete. Finally we show that most groups of the form $\Gamma(4, q, r; n)$, where I_{1323} is elliptic, are non-discrete.

Definition 2.1.1 *Let (p, q, r) be a triple of natural numbers. The (p, q, r)* ***triangle group*** *is the Coxeter group with presentation,*

$$\langle I_1, I_2, I_3 \mid I_1^2, I_2^2, I_3^2, (I_2I_3)^p, (I_3I_1)^q, (I_1I_2)^r \rangle.$$

These groups are called triangle groups since there is an obvious representation of (p, q, r) into $\mathrm{Isom}(\mathbf{S}^2)$, $\mathrm{Isom}(\mathbf{E}^2)$ or $\mathrm{Isom}(\mathbf{H}^2_{\mathbb{R}})$, where we identify the generators with reflections in the sides of a geodesic triangle with vertex angles π/p, π/q and π/r. When $1/p + 1/q + 1/r < 1$, we call the corresponding group a **hyperbolic triangle group** since the corresponding representation, which we denote ρ_π, is into $\mathrm{Isom}(\mathbf{H}^2_{\mathbb{R}}) = \mathbf{PO}(2, 1)$. In $\mathbf{H}^2_{\mathbb{R}}$ any two triangles with the same vertex angles are isometric, so for any hyperbolic triangle group, (p, q, r), the corresponding representation, ρ_π, is unique up to conjugation in $\mathbf{PO}(2, 1)$.

Proposition 2.1.2 $\rho_\pi : (p, q, r) \to \mathbf{PO}(2, 1)$ *described above is faithful. Moreover* $\rho_\pi(p, q, r) \subset \mathbf{PO}(2, 1)$ *is discrete and cocompact.*

We also allow p, q or $r = \infty$, then the corresponding triangle will have an ideal vertex and the related word in the Coxeter group will be sent by ρ_π to a parabolic element in $\mathbf{PO}(2, 1)$, although this extra piece of information will not be visible from the presentation. More detail on hyperbolic triangle groups and the geometry of the real hyperbolic plane in general can be found in most introductory books on hyperbolic geometry e.g. [1] or [13].

2.1.1 The deformation problem

Let (p, q, r) be a hyperbolic triangle group, i.e. $1/p+1/q+1/r < 1$, then we may also consider representations of (p, q, r) into larger Lie groups $G \supset \mathbf{PO}(2, 1)$ and ask if $\rho_\pi : (p, q, r) \to \mathbf{PO}(2, 1)$ fits into a larger family of representations $\rho_t : (p, q, r) \to G$. In this thesis we consider the extra representations into $G = \mathbf{PU}(2, 1)$, the group of holomorphic isometries of $\mathbf{H}^2_{\mathbb{C}}$.

The construction of representations of (p, q, r) into $\mathbf{PU}(2, 1)$ is analogous to the real hyperbolic case. We take a triangle in $\mathbf{H}^2_{\mathbb{C}}$ with the required vertex angles and identify the generators of the group with the complex involutions in the $\mathbb{C}$-lines spanned by the vertices of the triangle. Up to conjugation in $\mathbf{PU}(2, 1)$, the space of non-isomorphic (p, q, r) triangles in $\mathbf{H}^2_{\mathbb{C}}$ is one real dimensional. This leads naturally to a one real parameter family of non-conjugate representations of (p, q, r). We denote this parameter by t and the corresponding representation ρ_t. In section 2.3 we describe a parametrisation for these representations. At the moment it is sufficient to say that we define our parametrisation so that the image of (p, q, r) under ρ_π stabilises an $\mathbb{R}$-plane and the map ρ_π is essentially a representation into $\mathbf{PO}(2, 1)$ embedded into $\mathbf{PU}(2, 1)$ by inclusion. When $t \neq \pi$, the representation is not conjugate to a representation into $\mathbf{PO}(2, 1)$ and we call it a **deformed ($\mathbb{R}$-Fuchsian) triangle group**. More precisely a deformed triangle group is the image of an abstract Coxeter group under a specified representation into $\mathbf{PU}(2,1)$.

From this point on, unless stated otherwise, we assume that $p \leq q \leq r$.

Definition 2.1.3 *Let $G \subset \mathbf{PU}(2,1)$,*

- *If G stabilises an $\mathbb{R}$-plane, $\mathcal{R}$, and acts discretely on $\mathcal{R}$, we say G is **$\mathbb{R}$-Fuchsian**.*
- *If G stabilises an $\mathbb{C}$-line, $\mathcal{C}$, and acts discretely on $\mathcal{C}$, we say G is **$\mathbb{C}$-Fuchsian**.*

2.1.2 Groups of type A and type B

Given a (p,q,r) triangle group (with $p \leq q \leq r$), define the words

$$W_A = I_1 I_3 I_2 I_3, \quad W_B = I_1 I_2 I_3.$$

Theorem 2.1.4 *([30], [12]) Let $\mathcal{I} \subset \mathbb{R}$ be the set of values of t in the parameter space such that $\rho_t(W_A)$ and $\rho_t(W_B)$ are both non-elliptic. Then the set $\mathcal{I}$ is a closed interval. We call $\mathcal{I}$ the **critical interval**.*

This theorem was conjectured in [30]. It was proved in [12]. Although it is not explicitly stated in that paper it follows from the arguments used to prove the main result (which we restate as theorem 2.1.6 below).

Definition 2.1.5 *A group (p,q,r) is **type A** if the end points of $\mathcal{I}$ correspond to a representation where W_A is parabolic. Otherwise we say (p,q,r) are **type B**.*

Theorem 2.1.6 *[12] The group (p,q,r) is of type A if $p < 10$ and type B if $p > 13$.*

Definition 2.1.7 *A word in $\rho_t(p,q,r)$ has **genuine length** n if it is not conjugate in $\rho_t(p,q,r)$ to any shorter word or a power of a shorter word. For example $I_1I_3I_2I_3$ has genuine length 4 and $I_1I_3I_2I_3I_2I_1$ has genuine length 2, since it is conjugate to $(I_3I_2)^2$.*

Then we can reformulate the definition of the critical interval as follows.

Definition 2.1.8 *The critical interval $\mathcal{I}$ can be defined as the set of values of t such that all words of genuine length 3 and 4 are non-elliptic.*

2.1.3 History and background of deformed triangle groups

The study of deformed ($\mathbb{R}$-Fuchsian) triangle groups began with Goldman and Parker in [11], where they analysed deformed (∞, ∞, ∞) groups. They showed that there is an interval $\mathbf{I}$ contained in $\mathcal{I}$ for which $\rho_t(\infty, \infty, \infty)$ is discrete and faithful for all $t \in \mathbf{I}$. They conjectured that $\rho_t(\infty, \infty, \infty)$ is discrete and faithful if and only if $t \in \mathcal{I}$. In [32] Schwartz proved a stronger version of this conjecture namely $\rho_t(\infty, \infty, \infty)$ is discrete and faithful if and only if $t \in \mathcal{I}$. Moreover if $t \notin \mathcal{I}$ then $\rho_t(\infty, \infty, \infty)$ is not discrete. Schwartz's proof can be applied to all triples (p, q, r) with sufficiently large p, although Schwartz does not calculate a specific bound. Schwartz then made the more general conjecture.

Conjecture 2.1.9 *For any triple (p, q, r), the representation $\rho_t(p, q, r)$ is faithful and discrete (as a subgroup of $\mathbf{PU}(2, 1)$) for all $t \in \mathcal{I}$.*

The next deformed triangle groups to be studied in depth were $\rho_t(4, 4, \infty)$ by Justin Wyss-Galifent in his thesis [36]. He made the surprising discovery of 'extra' discrete, but non-faithful, representations *outside* of $\mathcal{I}$. These representations correspond to values of t where $\rho_t(I_1 I_3 I_2 I_3)$ is elliptic and of finite order.

Definition 2.1.10 *We define $\Gamma(p, q, r; n)$ to be the image of a representation with $t \notin \mathcal{I}$ where $\rho_t(I_1 I_3 I_2 I_3)$ is an elliptic word of order n.*

Remark: The 4-tuple $\Gamma(p, q, r; n)$ does not denote an abstract group, it denotes a matrix subgroup of $\mathbf{PU}(2, 1)$. The matrix subgroup is the image of a specific representation of (p, q, r) into $\mathbf{PU}(2, 1)$. Obviously we can write down an abstract presentation for the group, but it is important to keep in mind that the presentation corresponds to a specific matrix group up to conjugation in $\mathbf{PU}(2, 1)$.

There is an ambiguity in this notation since it may be possible that there are two or more values of t such that $\rho_t(I_1 I_3 I_2 I_3)$ has order n. We briefly discuss this in the section on non-standard deformed triangle groups.

We could consider values of t for which $\rho_t(I_1 I_3 I_2 I_3)$ is elliptic of infinite order. These groups are necessarily non-discrete and therefore not particularly interesting. There is also the question of groups of type B, i.e. groups where $\rho_t(I_1 I_2 I_3)$ is elliptic whilst $\rho_t(I_1I_3I_2I_3)$ is still loxodromic, these groups are more difficult to deal with since it is a lot more difficult to determine the behaviour of $\rho_t(I_1I_2I_3)$ than $\rho_t(I_1I_3I_2I_3)$ and we don't consider them in this thesis.

In this language, the extra discrete deformed triangle groups studied by Wyss-Galifent are $\Gamma(4,4,\infty;3)$, $\Gamma(4,4,\infty;4)$ and $\Gamma(4,4,\infty;\infty)$. Influenced by this discovery, Schwartz conjectured that complex hyperbolic hyperbolic triangle groups fall into two categories, depending on whether they are type A or type B.

Conjecture 2.1.11 *If (p,q,r) is of type A, then $\rho_t(p,q,r)$ is discrete and faithful for t inside the critical interval. In addition there is a countable (possibly infinite or zero) collection of points, t_m, outside the critical interval, such that the group $\Gamma(p,q,r;n) = \rho_{t_m}(p,q,r)$ is discrete but non-faithful. If (p,q,r) is of type B, then $\rho_t(p,q,r)$ is discrete if and only if t lies inside the critical interval.*

The next case to be studied were the deformed $(4,4,4)$ groups, In 2006, Deraux proved that $\Gamma(4,4,4;5)$ was a lattice by demonstrating that a Dirichlet domain for the group is bounded [6]. In 2003, Schwartz showed that the groups $\Gamma(4,4,4;n)$ for $n = 5,6,7,8,12,18$ are arithmetic [31]. In the same paper, the group $\Gamma(4,4,4;7)$ was studied in great depth. In [21] Parker analysed all discrete groups of the form $\Gamma(p,p,p;n)$ using a result of Conway and Jones [3] and found two counterexamples to conjecture 2.1.11, namely a deformed $(14,14,14)$ group and the group $\Gamma(18,18,18;18)$.

Recent results by Pratoussevitch [27] and Kamiya, Parker & Thompson [18] have shown that a large number of deformed (m,m,∞) groups are non-discrete.

The study of deformed triangle groups is strongly related to work done by Mostow, Deligne, Sauter and others in the 1980s. In 1980 Mostow discovered a number of lattices in $\mathbf{PU}(2,1)$ arising from monodromy groups of hypergeometric functions. Many of these lattices had surprising geometric properties, in the case of triangle groups, there is a family of these lattices that contain $\Gamma(p,p,p;p)$ groups as infinite index subgroups (or index 60 in the case $p = 5$). These groups were also studied by Livné's in his thesis [15] where they arise in the context of algebraic geometry. For an overview of this area and how it relates to complex hyperbolic geometry, see Parker's survey paper [23].

2.2 Notation

Deformed triangle groups are generated by three complex involutions I_1, I_2 and I_3. These involutions fix $\mathbb{C}$-lines which we denote $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ respectively. We write compositions of reflections, for example $I_1I_2I_1$ as I_{121} or simply 121. We call this **word notation** and such a string of numbers is called a **word**.

When a word is palindromic, the corresponding isometry is a complex involution, since it can be thought of as conjugate to one of the generating involutions. For example the line $I_1(\mathcal{C}_2) = \mathcal{C}_{121}$ is the $\mathbb{C}$-line fixed by the complex involution $I_{121} = I_1I_2I_1^{-1}$. The composition of two word isometries is simply the composition of the words, e.g.

$$I_{121}I_{131} = I_{121131}.$$

Since complex involution are of order two, i.e. $I_jI_j = Id$, in word notation we can use this condition to delete (or introduce) double letters in a word, so the above example would become,

$$I_{121}I_{131} = I_{1231}.$$

The other group relations, e.g $(I_{23})^p$ and $(I_{1323})^n = Id$, may be also used to simplify words.

We also use word notation to denote points lying in the intersection of two lines defined by words, we compose the two words and delete any repeated digits. For example, the point of intersection between the lines $\mathcal{C}_{2321232}$ and $\mathcal{C}_{121}$ is denoted $p_{2321232121}$ (or 2321232121 if there is no risk of confusion). Alternatively we can interpret this convention so that the point p_w is the unique fixed point of a regular elliptic isometry I_w. In this notation the vertices of the generating triangle are p_{12}, p_{23}, p_{31} (or $12, 23, 31$).

A word and its reverse therefore denote the same line or point, however in general this will not be true for words denoting isometries e.g. $I_{12} = (I_{21})^{-1} \neq I_{21}$. Clearly the fixed points of I_{12} and I_{21} are the same, so $p_{12} = p_{21}$.

It is easy to write down the action of any word in our group (i.e an isometry generated by I_1, I_2, I_3) on a point of $\mathbf{H}^2_{\mathbb{C}}$ denoted by a word. We simply conjugate the word corresponding to the point by the word corresponding to the isometry and use the group relations to simplify, for instance,

$$I_{1232}\left(p_{232121}\right) = (1232)(232121)(2321) = \underline{12322321}212321 = 212321 = p_{212321}$$

This notation introduces some ambiguity between isometries, $\mathbb{C}$-lines fixed by these isometries and points lying in the intersection of two $\mathbb{C}$-lines since all three may be denoted by words. However it will generally be clear from the context which of the three is being discussed and where further clarity is required we shall revert to I, $\mathcal{C}$, p notation for isometries, lines and points respectively.

2.3 The parameter space of $\rho_t(p,q,r)$

We now construct a representation for a general $\Gamma(p,q,r;n)$ triangle group and describe the deformation space for (p,q,r) triangle groups. We can choose a basis of $\mathbb{C}^{2,1}$ so that the polar vectors, $\mathbf{n}_i$ to the fixed complex lines of I_i are

$$\mathbf{n}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{n}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{n}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

satisfying

$$\langle \mathbf{n}_1, \mathbf{n}_1 \rangle = \langle \mathbf{n}_2, \mathbf{n}_2 \rangle = \langle \mathbf{n}_3, \mathbf{n}_3 \rangle = 2, \quad \langle \mathbf{n}_2, \mathbf{n}_1 \rangle = \rho, \quad \langle \mathbf{n}_3, \mathbf{n}_2 \rangle = \sigma, \quad \langle \mathbf{n}_1, \mathbf{n}_3 \rangle = \tau.$$

Then the Hermitian form is given by

$$H = \begin{bmatrix} 2 & \rho & \overline{\tau} \\ \overline{\rho} & 2 & \sigma \\ \tau & \overline{\sigma} & 2 \end{bmatrix}.$$

In order to have signature $(2,1)$ we must have $\det(H) < 0$. That is

$$\rho\sigma\tau + \overline{\rho\sigma\tau} - 2|\rho|^2 - 2|\sigma|^2 - 2|\tau|^2 + 8 < 0. \tag{2.1}$$

Let I_j be the complex involution in the $\mathbb{C}$-line orthogonal to $\mathbf{n}_j$. Then using formula (1.3) from chapter 1,

$$I_j(\mathbf{z}) = -\mathbf{z} + 2\frac{\langle \mathbf{z}, \mathbf{n}_j\rangle}{\langle \mathbf{n}_j, \mathbf{n}_j\rangle}\mathbf{n}_j$$

we have

$$I_1 = \begin{bmatrix} 1 & \rho & \overline{\tau} \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad I_2 = \begin{bmatrix} -1 & 0 & 0 \\ \overline{\rho} & 1 & \sigma \\ 0 & 0 & -1 \end{bmatrix}, \quad I_3 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ \tau & \overline{\sigma} & 1 \end{bmatrix}.$$

We can calculate 1-eigenvectors for I_iI_j , which we denote $\mathbf{v}_{ij}$,

$$\mathbf{v}_{12} = \begin{bmatrix} \rho\sigma - 2\overline{\tau} \\ \overline{\rho\tau} - 2\sigma \\ 4 - |\rho|^2 \end{bmatrix}, \quad \mathbf{v}_{23} = \begin{bmatrix} 4 - |\sigma|^2 \\ \sigma\tau - 2\overline{\rho} \\ \overline{\rho\sigma} - 2\tau \end{bmatrix}, \quad \mathbf{v}_{31} = \begin{bmatrix} \overline{\sigma\tau} - 2\rho \\ 4 - |\tau|^2 \\ \tau\rho - 2\overline{\sigma} \end{bmatrix}. \tag{2.2}$$

These vectors are lifts of the fixed points of I_{12}, I_{23}, I_{31} in $\mathbf{H}^2_{\mathbb{C}}$. The corresponding points p_{12}, p_{23}, p_{31} are the vertices of the triangle which defines the $\mathbb{C}$-lines that are fixed by the generating reflections.

The traces of I_{ij} are

$$\operatorname{tr}(I_1I_2) = |\rho|^2 - 1, \quad \operatorname{tr}(I_2I_3) = |\sigma|^2 - 1, \quad \operatorname{tr}(I_3I_1) = |\tau|^2 - 1.$$

By fixing

$$|\sigma| = 2\cos(\pi/p), \quad |\tau| = 2\cos(\pi/q), \quad |\rho| = 2\cos(\pi/r),$$

we ensure that I_{12}, I_{23} and I_{31} have the required order.

The trace of the triple product I_{123} is

$$\operatorname{tr}(I_1I_2I_3) = \rho\sigma\tau - |\rho|^2 - |\sigma|^2 - |\tau|^2 + 3. \tag{2.3}$$

Combining this with (2.1) we see that

$$\operatorname{Re}\big(\operatorname{tr}(I_1I_2I_3)\big) < -1.$$

Definition 2.3.1 *After fixing $|\sigma|$, $|\tau|$, $|\rho|$, there is, up to conjugation in* $\mathbf{PU}(2,1)$, *a one parameter family of deformed (p,q,r)-triangle groups. The parameter, t, is the argument of $\rho\sigma\tau$.*

We can always normalise so that the imaginary part of $\rho\sigma\tau$ is positive (applying an anti-holomorphic involution if necessary) so that the argument of $\rho\sigma\tau$ always lies in the interval $[0,\pi]$.

The angular invariant

We can also understand the deformation parameter of a triangle group in terms of the generalised angular invariant

Definition 2.3.2 *Let $\boldsymbol{n}_1$, $\boldsymbol{n}_2$ and $\boldsymbol{n}_3$ be three polar vectors defining a complex hyperbolic triangle, then the* ***generalised angular invariant*** *of $\boldsymbol{n}_1$, $\boldsymbol{n}_2$, $\boldsymbol{n}_3$ is*

$$\mathbb{A}(\boldsymbol{n}_1,\boldsymbol{n}_2,\boldsymbol{n}_3)=\arg(-\langle\boldsymbol{n}_3,\boldsymbol{n}_2\rangle\langle\boldsymbol{n}_2,\boldsymbol{n}_1\rangle\langle\boldsymbol{n}_1,\boldsymbol{n}_3\rangle) \tag{2.4}$$

Remark: It follows from a quick calculation that the angular invariant for a deformed triangle is $\pi-\arg(\rho\sigma\tau)$. When the angular invariant $\mathbb{A}(\mathbf{n}_1,\mathbf{n}_2,\mathbf{n}_3)$ is equal to 0, $\rho\sigma\tau$ is real and the corresponding deformed triangle group is $\mathbb{R}$-Fuchsian. This is equivalent to the three vertices of the corresponding triangle, p_{12}, p_{23} and p_{31}, all lying in the same $\mathbb{R}$-plane and the corresponding presentation being into $\mathbf{PO}(2,1)\subset\mathbf{PU}(2,1)$.

Alternative notations

There are a variety of different notations used to describe deformed triangle groups, we have followed the notation from [21]. We have also followed Schwartz's convention that $\mathrm{ord}(I_2I_3)=p$, $\mathrm{ord}(I_3I_1)=q$, $\mathrm{ord}(I_1I_2)=r$, $\mathrm{ord}(I_1I_3I_2I_3)=n$ and $p\le q\le r$. This leads to the slightly awkward situation where $|\sigma|\le|\tau|\le|\rho|$.

We can convert our ρ, σ, τ notation into the notation used by Pratoussevitch in [26] as follows

$$|\sigma|=2r_1,\ \ |\tau|=2r_2,\ \ |\rho|=2r_3 \text{ and } \arg(\rho\sigma\tau)=\alpha.$$

In [32] Schwartz uses similar notation, however the indices are offset by one i.e Schwartz's r_i is the same as Pratoussevitch's r_{i+1}.

After fixing $|\rho|$, $|\sigma|$, $|\tau|$, the deformed (p,q,r)-triangle group corresponding to $t=\pi$, fixes an $\mathbb{R}$-plane. As we decrease t the word I_{1323} becomes 'more elliptic'. At the point

$$t_0 = \arccos\left(\frac{|\rho|^2+|\sigma\tau|^2-4}{2|\rho\sigma\tau|}\right)$$

the word I_{1323} becomes parabolic. For all $t<t_0$ (respectively $t>t_0$), I_{1323} is loxodromic (respectively elliptic). At the point

$$t_1 = \arccos\left(\frac{|\rho|^2+|\sigma|^2+|\tau|^2-4}{|\rho\sigma\tau|}\right)$$

the group becomes degenerate. These values are essentially the same as c_A and c_∞ from Section 11 of [26] (and lemma 4.2 of [32]).

We split the parameter space into three sub-intervals $[0,t_1]$, (t_1,t_0) and $[t_0,\pi]$. For $t\in[0,t_1]$ all the deformed triangle groups are degenerate. If (p,q,r) is of type A, then $t\in[t_0,\pi]$ is precisely the critical interval $\mathcal{I}$. We are interested in deformed triangle groups corresponding to values of t in the middle interval (t_1,t_0). If t lies in this interval, the group $\rho_t(p,q,r)$ is non-degenerate and I_{1323} is elliptic.

We now show how to choose a value for $t=\arg(\rho\sigma\tau)$ to ensure that I_{1323} has order n.

$$\mathrm{tr}(I_1I_2I_1I_3)=|\rho\tau-\overline{\sigma}|^2-1,\quad \mathrm{tr}(I_2I_3I_2I_1)=|\rho\sigma-\overline{\tau}|^2-1,\quad \mathrm{tr}(I_3I_1I_3I_2)=|\sigma\tau-\overline{\rho}|^2-1.$$

Suppose that

$$|\rho\tau-\overline{\sigma}|=2\cos(\pi/l),\quad |\rho\sigma-\overline{\tau}|=2\cos(\pi/m),\quad |\sigma\tau-\overline{\rho}|=2\cos(\pi/n).$$

Squaring both sides we obtain,

$$\begin{aligned}\rho\sigma\tau+\overline{\rho\sigma\tau} &= 16\cos^2(\pi/p)\cos^2(\pi/r)+4\cos^2(\pi/q)-4\cos^2(\pi/l)\\ &= 16\cos^2(\pi/p)\cos^2(\pi/q)+4\cos^2(\pi/r)-4\cos^2(\pi/m)\\ &= 16\cos^2(\pi/q)\cos^2(\pi/r)+4\cos^2(\pi/p)-4\cos^2(\pi/n).\end{aligned}\tag{2.5}$$

Corollary 2.3.3 *Let $\Gamma(p, q, r; n)$ be a complex hyperbolic triangle group and define ρ, σ, τ as above. Then,*

$$2\mathrm{Re}\,(\rho\sigma\tau) = -2\cos(2\pi/n) - 2 + |\rho|^2 + |\sigma\tau|^2 \tag{2.6}$$

Alternatively, the corresponding value of t is

$$t = \arccos\left(\frac{-2\cos(2\pi/n) - 2 + |\rho|^2 + |\sigma\tau|^2}{2|\rho\sigma\tau|}\right). \tag{2.7}$$

Lemma 2.3.4 *Let $A \in \mathbf{PU}(2,1)$, if $\mathrm{tr}(A)$ is real and contained in $(-1, 3)$, then A is regular elliptic, $\mathrm{tr}(A) = 1 + 2\cos(\psi)$ and A has finite order if and only if ψ is a rational multiple of π.*

Proof: Since $\mathrm{tr}(A) \in (-1, 3)$, it follows from Goldman's trace formula 1.2.12 that A is regular elliptic. As A is regular elliptic it has three eigenvalues of unit modulus. The product of these eigenvalues is 1 and the sum is real. We can therefore write these eigenvalues as $e^{i\theta}$, $e^{i\phi}$ and $e^{-i\theta-i\phi}$. Considering the imaginary part of $\mathrm{tr}(A)$ we get,

$$\begin{aligned} 0 = \mathrm{Im}(\mathrm{tr}(A)) &= \sin(\theta) + \sin(\phi) - \sin(\theta + \phi) \\ &= 4\sin(\theta/2)\sin(\phi/2)\sin(\theta/2 + \phi/2). \end{aligned}$$

At least one of θ, ϕ or $\theta + \phi$ is equal to an integer multiple of π. So at least one of the eigenvalues is equal to 1. Then it follows that the other eigenvalues are $e^{i\psi}$ and $e^{-i\psi}$ and A has trace $1 + 2\cos(\psi)$. If ψ is a rational multiple of π, then A has finite order, otherwise A has infinite order. □

Non-standard deformed triangle groups

We have defined $\Gamma(p, q, r; n)$ as the group corresponding to the value of t that has $\mathrm{tr}(I_{1323}) = 1 + \cos(2\pi/n)$, this forces I_{1323} to have order n. In general there will be more than one value of t for which, in the corresponding deformed triangle group, I_{1323} has order n. These are the groups where $\mathrm{tr}(I_{1323}) = 1 + \cos(2m\pi/n)$, where m and n are co-prime and $t \in (t_1, t_0)$. For example the group $\Gamma(4, 4, 4; 70)$ corresponds to $t = 1.2079\ldots$, but there are 5 other values of t in the interval (t_1, t_0) that will also produce a deformed $(4, 4, 4)$ group with $\mathrm{ord}(I_{1323}) = 70$. They are

t	$\mathrm{tr}(I_{1323})$
$1.2079\ldots$	$1+2\cos(2\pi/70)$
$1.1957\ldots$	$1+2\cos(6\pi/70)$
$1.0896\ldots$	$1+2\cos(18\pi/70)$
$1.0328\ldots$	$1+2\cos(22\pi/70)$
$0.9665\ldots$	$1+2\cos(26\pi/70)$
$0.8075\ldots$	$1+2\cos(34\pi/70)$

Note that for $(4,4,4)$ the interval $(t_1,t_0)=(0.7859\ldots,1.2094\ldots)$.

For the moment we ignore groups where $\mathrm{tr}(I_{1323})\neq 1+2\cos(2\pi/n)$, since there are reasonably convincing arguments that, unless n is large and m small, these groups will not be discrete (see section 2.5.1). We refer to these groups as **non-standard deformed triangle groups** and we denote them by $\Gamma(p,q,r;n/m)$ where $\mathrm{tr}(I_{1323})=1+2\cos(2m\pi/n)$.

2.4 Identifications between $\Gamma(p,q,r;n)$

In this section, we describe a family of identifications between different deformed triangle groups. In this context, the identification consists of choosing a different generating set of matrices for a deformed triangle group and rewriting the group relations in terms of these new generators to change one deformed triangle group into another one.

The group $(p,q,r;n)$ is generated by three order two complex reflections I_1, I_2 and I_3. We define ι_1 to be the involution of groups that acts on the generating set (I_1,I_2,I_3) as follows,

$$\iota_1(I_1)=I_1,\quad \iota_1(I_2)=I_{121},\quad \iota_1(I_3)=I_3.$$

i.e. we conjugate the second generator by the first. The map ι_1 extends to the rest of the group the obvious way. Since I_1 has order two it is clear that ι_1^2 acts trivially. There are two analogous group involutions ι_2 and ι_3, which we define as follows

$$\iota_2(I_1,I_2,I_3)=(I_1,I_2,I_{232})\ \text{ and }\ \iota_3(I_1,I_2,I_3)=(I_{313},I_2,I_3).$$

Recall, we apply these group involutions to the matrix group $\rho_t(p,q,r)$ rather than to the abstract group (p,q,r). We are relabelling the matrices that generate $\rho_t(p,q,r) \subset \mathbf{PU}(2,1)$ rather than generators of an abstract group.

Let $A_1 = \iota_i(I_1)$, $A_2 = \iota_i(I_2)$ and $A_3 = \iota_i(I_3)$ for some ι_i. Then ι_i produces a representation for another triangle group presentation of Γ in terms of A_j, namely

$$\langle A_1, A_2, A_3 \mid A_i^2, A_{23}^{p'}, A_{31}^{q'}, A_{12}^{r'}, A_{1323}^{n'} \rangle.$$

This is the same matrix subgroup of $\mathbf{PU}(2,1)$ that we started with. All the ι do is change the generating set and then we must find the required relations in terms of these new generators. This produces an identification $\Gamma(p,q,r;n) \sim \Gamma(p',q',r';n')$. In general there is no reason to expect A_{23}, A_{31}, A_{12} or A_{1323} to be finite order elliptic or parabolic. When one or more of the A_{ij} is loxodromic we have a **generalised deformed triangle group**, that is a group generated by three order two reflections in $\mathbb{C}$-lines that do not pairwise intersect in $\mathbf{H}^2_{\mathbb{C}}$ to form a triangle.

It is also possible that the new group is a non-standard deformed triangle group, to determine this we have to check the trace of the words A_{ij} and A_{ijkj} and not just their order. For small values of n' is is very unlikely that the group will be non-standard, recall from the $\Gamma(4,4,4;70)$ example there were only 5 possible non-standard groups and for $n < 11$ there are no non-standard $\Gamma(4,4,4;n)$ groups. These trace calculations are straightforward and we omit them unless they lead to a deformed triangle group.

Lemma 2.4.1 *The involution ι_3 identifies the deformed triangle groups $\Gamma(p,q,r;n)$ and $\Gamma(p,q,n;r)$.*

Proof: Let (I_1, I_2, I_3) be a triple of reflections that generate $\Gamma(p,q,r;n)$ with the necessary relations, I_{23}^p, I_{31}^q, I_{12}^r and I_{1323}^n. Then the involution ι_3 sends this set to $(A_1 = I_{313}, A_2 = I_2, A_3 = I_3)$. In these generators, the words we use to specify a unique deformed triangle group are,

$$\begin{aligned}
A_{23} &= A_2A_3 = I_2I_3 = I_{23}, \\
A_{31} &= A_3A_1 = I_3I_3I_1I_3 = I_1I_3 = I_{31}^{-1}, \\
A_{12} &= A_1A_2 = I_3I_1I_3I_2 = I_3(I_{1323})I_3, \\
A_{1323} &= A_1A_3A_2A_3 = I_3I_1I_3I_3I_2I_3 = I_3(I_{12})I_3.
\end{aligned}$$

Then the group generated by the A_i has the following partial presentation,

$$\langle A_1, A_2, A_3 \mid A_i^2, A_{23}^p, A_{31}^q, A_{12}^n, A_{1323}^r \rangle.$$

We now have to check the trace of A_{1323} to ensure that group is not non-standard, once this is done we have uniquely determined the representation in to $\mathbf{PU}(2, 1)$. There may other group relations not shown in the partial presentation, but these do not change the specific deformed triangle group. $\square$

The involution ι_3 has this special property due to the appearance of I_{1323} in the relations that we chose to classify triangle groups. For ι_1 and ι_2 the situation is more complicated (see the tables in chapter 5 for details).

We can think of these maps as order 2 identifications between deformed triangle groups. Applying ι_i repeatedly will produce new generating sets and new presentations for the group. There is no reason to expect two different sequences of involutions to produce the same generating set so, in general, the graph of all possible generating sets is the valency three tree, part of which is shown in figure 2.1.

Each vertex of the tree corresponds to a triple of words in the generating set which give a presentation of the triangle group, so this could instead be thought of as a tree of isomorphisms between triangle groups. Closer analysis shows that the relations of the original group will cause many vertices to collapse to the same point so this graph is no longer a tree.

For the sake of clarity we work though some concrete examples.

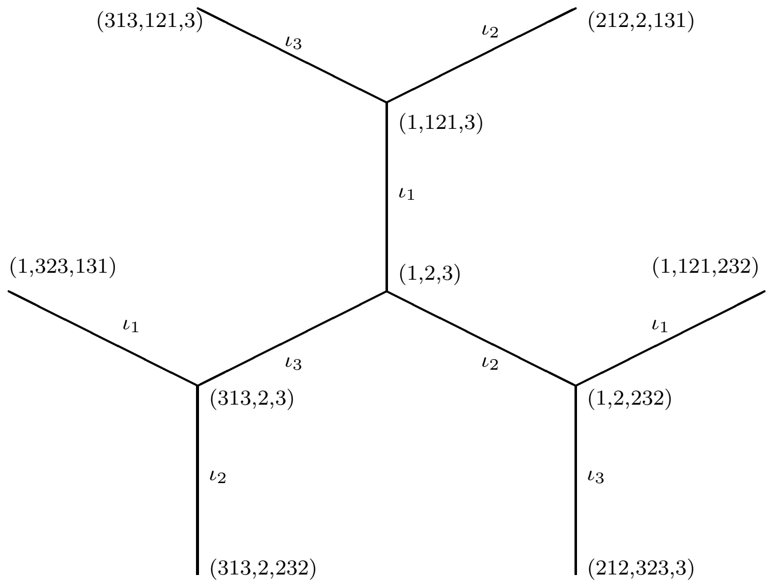

Figure 2.1: Graph of generating sets

$\Gamma(4,4,4;5)$

$\Gamma(4,4,4;5) = \langle I_1, I_2, I_3 | I_i^2, I_{ij}^4, I_{1323}^5 \rangle$. Under ι_1 this group is sent to

$$\langle \iota_1(I_1), \iota_1(I_2), \iota_1(I_3) | (\iota_1(I_i))^2, (\iota_1(I_{23}))^5, (\iota_1(I_{31}))^4, (\iota_1(I_{12}))^4, (\iota_1(I_{1323}))^5 \rangle.$$

So we conclude that $\iota_1 : \Gamma(4,4,4;5) \leftrightarrow \Gamma(5,4,4;5)$ is an identification between these two deformed triangle groups. Since ι_1 is an involution, ι_1^2 acts trivially and preserves the original presentation. The other involutions give the following identifications

$$\iota_2 : \Gamma(4,4,4;5) \leftrightarrow \Gamma(4,5,4;5),$$
$$\iota_3 : \Gamma(4,4,4;5) \leftrightarrow \Gamma(4,4,5;4).$$

Although these groups have different $\Gamma(p,q,r;n)$ presentations, when we permute the generating reflections to put them into the standard form (i.e. $p \le q \le r$) we see that they are the same group, namely $\Gamma(4,4,5;4)$.

The group relations for $\Gamma(4,4,4;5)$ cause the corresponding tree to reduce to a finite graph. To see this, notice that $\iota_3\iota_1(I_1, I_2, I_3) = (I_{313}, I_{121}, I_3)$ and $\iota_1\iota_3(I_1, I_2, I_3) =$

(I_1, I_{323}, I_{131}), conjugating the second triple by I_{13} produces $(I_{13131}, I_{121}, I_{1313131})$. In $\Gamma(4,4,4;5)$, $I_{31313} = I_{131}$, so the vertices corresponding to ι_{13} and ι_{31} collapse to a single vertex. There are similar relations related to I_{12}^4 and I_{23}^4. This has only used the fact that we're in a $\Gamma(4,4,4;n)$ group, so the same collapsing will occur for all n. Now we list the groups produced by ι_{12}, ι_{23} and ι_{31}.

$$\iota_{12} : \Gamma(4,4,4;5) \leftrightarrow \Gamma(5,5,4;6),$$
$$\iota_{23} : \Gamma(4,4,4;5) \leftrightarrow \Gamma(4,5,5;4),$$
$$\iota_{31} : \Gamma(4,4,4;5) \leftrightarrow \Gamma(5,4,5;4).$$

The next level of the graph will only consist of 3 points corresponding to $\iota_{312} = \iota_{321}$, $\iota_{123} = \iota_{132}$ and $\iota_{231} = \iota_{213}$. These involutions send (I_1, I_2, I_3) to the following generating sets

$$\iota_{312}(I_1, I_2, I_3) = (I_{212}, I_{31213}, I_3),$$
$$\iota_{123}(I_1, I_2, I_3) = (I_1, I_{323}, I_{12321}),$$
$$\iota_{231}(I_1, I_2, I_3) = (I_{23132}, I_2, I_{131})$$

and from this we can work out the new group presentations from these generators, namely

$$\iota_{312} : \Gamma(4,4,4;5) \leftrightarrow \Gamma(5,5,6;4),$$
$$\iota_{123} : \Gamma(4,4,4;5) \leftrightarrow \Gamma(6,5,5;5),$$
$$\iota_{231} : \Gamma(4,4,4;5) \leftrightarrow \Gamma(5,6,5;5).$$

After permuting indices to put the group in the standard form, we see that all these groups are isomorphic to $\Gamma(5,5,6;4)$. At this point, the graph terminates, in the sense that any longer word in ι_i takes us to a generating set which can be reduced (using the group relations) to another generating set arising from a shorter ι word. In particular this uses the $I_{ijik}^5 = Id$ relation from the original group, for general $\Gamma(4,4,4;n)$ groups the graph will be larger. The graph of presentations for $\Gamma(4,4,4;5)$ after collapsing is shown in figure 2.2. After putting the groups in the standard form, they all have one of the following presentations

$$\Gamma(4,4,4;5),\ \Gamma(4,4,5;4),\ \Gamma(4,5,5;4),\ \Gamma(5,5,6;4).$$

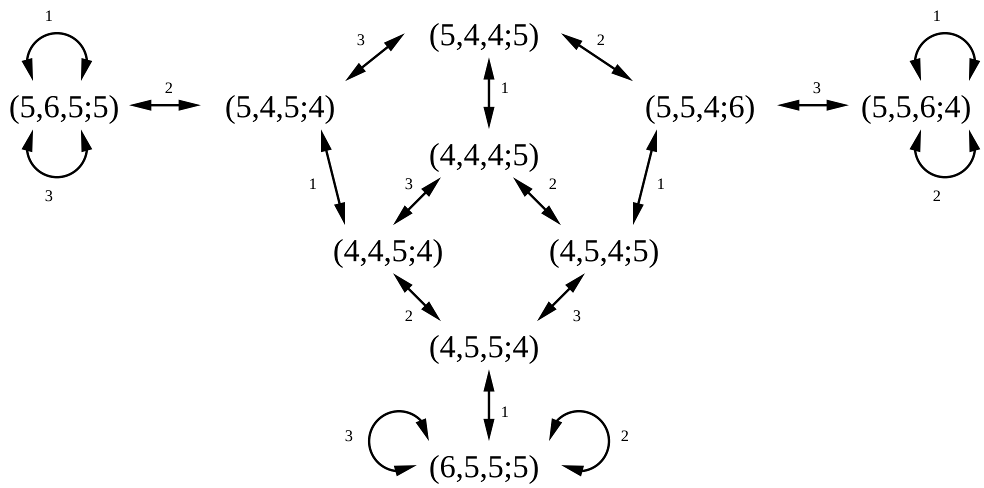

Figure 2.2: Graph of identifications for $\Gamma(4, 4, 4; 5)$

This is a rather special case, in that no sequence of ι maps send the group to a generalised triangle group. The only triangle groups with this property appear to be lattices (and degenerate, finite groups).

Corollary 2.4.2 *These are the only triangle groups isomorphic, under some ι_w, to $\Gamma(4, 4, 4; 5)$.*

$\Gamma(3, 3, 4; n)$ **(with $n > 7$)**

We now work through a non-lattice example to show how generalised triangle groups occur. Following the process as before, we produce the graph of isometries shown in figure 2.3. The generalised triangle groups are the groups contained in boxes, we terminate the graph at these groups since there is no obvious way of extending the notation and we believe the graph will be infinite. If the group were a lattice, at these points the edges of the graphs would form loops, i.e. two of the ι isomorphisms would send the group to itself (as in the $\Gamma(4, 4, 4; 5)$ graph, figure 2.2) This suggests an approach to find deformed triangle groups that are lattices, namely find groups where $\operatorname{ord}(23) = \operatorname{ord}(2131)$ and $\operatorname{ord}(31) = \operatorname{ord}(3212)$, see the tables in chapter 5. After putting the groups in the standard form, these deformed triangle groups all have one of the following presentations.

$$\Gamma(3, 3, 4; n),\ \Gamma(3, 3, n; 4),\ \Gamma(3, 4, n; 3),\ \Gamma(4, n, n; 3).$$

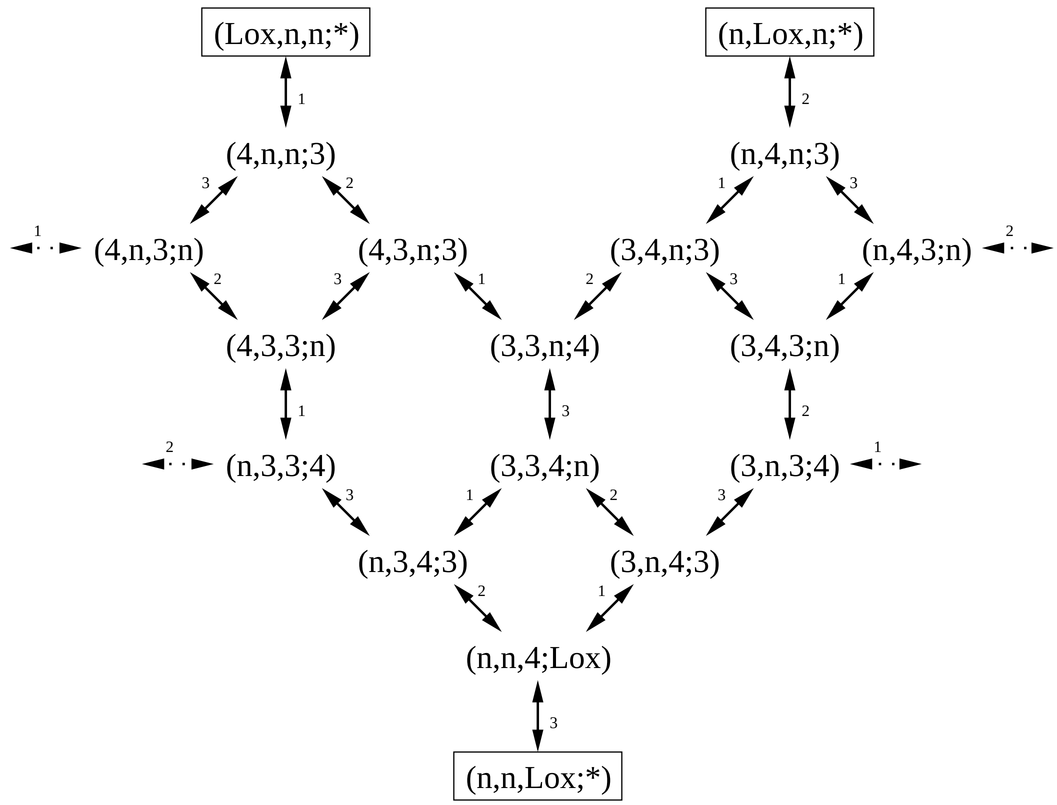

Figure 2.3: Graph of identifications for $\Gamma(3, 3, 4; n)$

$\Gamma(4, p, p; n)$

Proposition 2.4.3 *If $p < n$ (respectively $p > n$), then the deformed triangle groups $\Gamma(4, p, p; n)$, $\Gamma(4, p, n; p)$ (respectively $\Gamma(4, n, p; n)$) and $\Gamma(4, n, n; p)$ are identified.*

Remark: With sufficient care and decent notation, it should be possible to keep track of these groups as ι maps them between generalised triangle groups, and potentially use discreteness results from chapter 2 of [36].

Definition 2.4.4

$$\boldsymbol{K} := \mathrm{Re}\,(\rho\sigma\tau) - |\sigma|^2 - |\tau|^2 - |\rho|^2,$$

$$\boldsymbol{L} := |\sigma\tau\rho|^2 - (\mathrm{Re}\,(\rho\sigma\tau))^2 = \mathrm{Im}\,(\rho\sigma\tau)^2.$$

Remark: The determinant of the Hermitian form H is $\mathbf{K} + 4$, so H has signature $(2, 1)$, and $\Gamma(p, r, q; n)$ corresponds to a complex hyperbolic triangle group, if and only if $\mathbf{K} < -4$. If $\mathbf{K} = -4$ we say the group is degenerate.

Lemma 2.4.5 *The quantities* $\boldsymbol{K}$ *and* $\boldsymbol{L}$ *are fixed under the identifications* ι_i.

PROOF: We only show this for ι_1, the other cases are essentially the same. Assume we have a $\Gamma(p,q,r;n)$ group with associated ρ, σ, τ. Let $\Gamma(p',q',r';n')$ be the image of $(p,q,r;n)$ under ι_1 with the associated parameters ρ', σ', τ'. Under ι_1 there is the following identification of words $(I_{23}, I_{31}, I_{12}, I_{1213}) \leftrightarrow (I_{1213}, I_{31}, I_{21}, I_{23})$. Using the trace formulae we see that $|\sigma'| = |\rho\tau - \overline{\sigma}|$, $|\tau'| = |\tau|$, $|\rho'| = |\rho|$ and $|\rho'\tau' - \overline{\sigma'}| = |\sigma|$. These relations imply that $\mathrm{Re}\,(\rho'\sigma'\tau') = -\mathrm{Re}\,(\rho\sigma\tau) + |\rho\tau|^2$. Then

$$\begin{aligned}
2\mathbf{K} - 2\mathbf{K}' =& 2\mathrm{Re}\,(\rho\sigma\tau) - 2|\sigma|^2 - 2|\tau|^2 - 2|\rho|^2 - 2\mathrm{Re}\,(\rho'\sigma'\tau') + 2|\sigma'|^2 + 2|\tau'|^2 + 2|\rho'|^2 \\
=& |\rho\tau|^2 + |\sigma|^2 - |\sigma'|^2 - 2|\sigma|^2 - 2|\tau|^2 - 2|\rho|^2 - |\rho'\tau'|^2 - |\sigma'|^2 + |\sigma|^2 + 2|\sigma'|^2 \\
& + 2|\tau'|^2 + 2|\rho'|^2 \\
=& |\rho\tau|^2 - 2|\tau|^2 - 2|\rho|^2 - |\rho'\tau'|^2 + 2|\tau'|^2 + 2|\rho'|^2 \\
=& 0.
\end{aligned}$$

Hence $\mathbf{K} = \mathbf{K}'$. Then

$$\begin{aligned}
\mathbf{L} - \mathbf{L}' &= |\sigma\tau\rho|^2 - (\mathrm{Re}\,(\rho\sigma\tau))^2 - |\sigma'\tau'\rho'|^2 - (\mathrm{Re}\,(\rho'\sigma'\tau'))^2 \\
&= |\tau\rho|^2(|\sigma|^2 - |\sigma'|^2) - (\mathrm{Re}\,(\rho\sigma\tau))^2 + (\mathrm{Re}\,(\rho\sigma\tau))^2 - 2|\tau\rho|^2\mathrm{Re}\,(\rho\sigma\tau) + |\tau\rho|^4 \\
&= |\tau\rho|^2(|\sigma|^2 - |\sigma'|^2 - 2\mathrm{Re}\,(\rho\sigma\tau) + |\tau\rho|^2) \\
&= |\tau\rho|^2(|\sigma|^2 - |\sigma'|^2 - |\sigma|^2 - |\tau\rho|^2 + |\sigma'|^2 + |\tau\rho|^2) \\
&= 0.
\end{aligned}$$

Hence $\mathbf{L} = \mathbf{L}'$. □

Remark: The terms $\mathbf{K}$ and $\mathbf{L}$ individually are not enough to distinguish deformed triangle groups. For example $\Gamma(4,4,4;6)$ and $\Gamma(4,4,\infty;3)$ both have $\mathbf{K} = -4.5$, but since they have different values for $\mathbf{L}$ they a not identified via ι.

Conjecture 2.4.6 *Two deformed triangle groups can be identified if and only if they have the same* $\boldsymbol{K}$ *and* $\boldsymbol{L}$.

2.5 Discrete groups

In this section we describe some technical results which allow us to quickly determine the discreteness of some groups, then we use these results to prove our main result, theorem 2.5.6.

Theorem 2.5.1 *Let $G = \Gamma(p, q, r; n)$ be a non-degenerate group with $p, q, r, n \in \{3, 4, 6, \infty\}$. Then G is discrete.*

PROOF: Converting theorem 9 of [26] into ρ, σ, τ notation, the trace of each element of the group is an integer polynomial in the following variables

$$|\sigma|^2,\ |\tau|^2,\ |\rho|^2,\ \rho\sigma\tau,\ \overline{\rho\sigma\tau}. \tag{2.8}$$

Using the presentation described above and lemma 2.6, it is clear, for $\{p, q, r, n\} \subset \{3, 4, 6, \infty\}$, that $|\rho|^2$, $|\sigma|^2$, $|\tau|^2$ and $2\mathrm{Re}\,(\rho\sigma\tau)$ are integers. There are two possibilities, either $\mathrm{Re}\,(\rho\sigma\tau) = m \in \mathbb{Z}$ or $\mathrm{Re}\,(\rho\sigma\tau) = m/2$ for some odd $m \in \mathbb{Z}$.

If $\mathrm{Re}\,(\rho\sigma\tau) = m \in \mathbb{Z}$, then $\mathbf{L} = \mathrm{Im}^2(\rho\sigma\tau) = |\rho\sigma\tau|^2 - \mathrm{Re}^2(\rho\sigma\tau)$ is also an integer. So $\rho\sigma\tau = m + i\sqrt{\mathbf{L}}$ and $\overline{\rho\sigma\tau} = m - i\sqrt{\mathbf{L}}$. By Pratoussevitch's result, the trace of any word in the group can be written as an integer polynomial in $m + i\sqrt{\mathbf{L}}$, so the trace of every word in $\Gamma(p, q, r; n)$ lies in the ring $\mathbb{Z}[i\sqrt{\mathbf{L}}]$. This is a discrete ring, so the group is not dense in $\mathbf{PU}(2, 1)$. In addition there is no point in $\mathbf{H}^2_{\mathbb{C}}$ invariant under the action of the group, then by corollary 4.51 of [2], the group is discrete.

If $\mathrm{Re}\,(\rho\sigma\tau) = m/2$ for m odd, then $\mathbf{L} = \mathrm{Im}^2(\rho\sigma\tau) = l/4$. We can express l in terms of $|\rho|^2$, $|\sigma|^2$, $|\tau|^2$ and m as follows, $l = 4|\rho\sigma\tau|^2 - m^2$, in particular, l is congruent to 3 modulo 4. Then $\rho\sigma\tau = \frac{m}{2} + \frac{i\sqrt{l}}{2}$ and the trace of any word can be written as some element of $\mathbb{Z}[(1 + i\sqrt{l})/2]$. By the same argument as the previous case, the group is not dense in $\mathbf{PU}(2, 1)$ and therefore discrete. □

Corollary 2.5.2 *Identifying groups via ι, the following (non-degenerate) deformed triangle groups are discrete $\Gamma(3, 3, 4; \infty)$, $\Gamma(3, 3, 6; 6)$, $\Gamma(3, 3, 6; \infty)$, $\Gamma(3, 3, \infty; \infty)$, $\Gamma(3, 4, 4; 6)$, $\Gamma(3, 4, 4; \infty)$, $\Gamma(3, 4, 6; 6)$, $\Gamma(3, 4, 6; \infty)$, $\Gamma(3, 4, \infty; \infty)$, $\Gamma(3, 6, 6; 6)$, $\Gamma(3, 6, 6; \infty)$, $\Gamma(3, \infty, \infty; \infty)$, $\Gamma(4, 4, 4; 6)$, $\Gamma(4, 4, 4; \infty)$, $\Gamma(4, 4, 6; 6)$, $\Gamma(4, 4, 6; \infty)$,*

$\Gamma(4,4,\infty;\infty)$, $\Gamma(4,6,6;6)$, $\Gamma(4,6,6;\infty)$, $\Gamma(3,\infty,\infty;\infty)$, $\Gamma(6,6,6;6)$, $\Gamma(6,6,6;\infty)$, $\Gamma(6,6,\infty;\infty)$, $(6,\infty,\infty;\infty)$.

Lemma 2.5.3 (Jørgensen's inequality [14]) *Let $A \in \mathbf{SU}(2,1)$ be a regular elliptic isometry of order $n \geq 7$ that preserves a Lagrangian plane (i.e. $\mathrm{tr}(A)$ is real). Suppose that A fixes a point $z \in \boldsymbol{H}^2_{\mathbb{C}}$. Let B be any element of $\mathrm{PU}(2,1)$ with $B(z) \neq z$. If*

$$\cosh\left(\frac{d(B(z),z)}{2}\right)\sin\left(\frac{\pi}{n}\right) < \frac{1}{2}, \tag{2.9}$$

then $\langle A, B\rangle$ is not discrete and consequently any group containing A and B is not discrete.

Corollary 2.5.4 *Let $G = \Gamma(p,q,r;n)$ with $p \leq q \leq r$. Let ρ, σ, τ be defined as in section 2.3. If $\mathrm{ord}(I_{12}) = r \geq 7$ and*

$$\left(\rho\sigma\tau + \overline{\rho\sigma\tau} - 2|\sigma|^2 - 2|\tau|^2 - |\rho|^2 + 4\right)^2 < 4 - |\rho|^2 \tag{2.10}$$

then Γ is not discrete.

If $\mathrm{ord}(I_{23}) = p \geq 7$ and

$$\left(\rho\sigma\tau + \overline{\rho\sigma\tau} - |\sigma|^2 - 2|\tau|^2 - 2|\rho|^2 + 4\right)^2 < 4 - |\sigma|^2 \tag{2.11}$$

then Γ is not discrete.

If $\mathrm{ord}(I_{31}) = q \geq 7$ and

$$\left(\rho\sigma\tau + \overline{\rho\sigma\tau} - 2|\sigma|^2 - |\tau|^2 - 2|\rho|^2 + 4\right)^2 < 4 - |\tau|^2 \tag{2.12}$$

then G is not discrete.

Proof: Set $I_{12} = A$, $I_3 = B$ and $z = \mathbf{v}_{12}$ (fixed point of I_{12}) in lemma 2.5.3. Then (2.9) becomes

$$\left|\frac{\langle I_3(\mathbf{v}_{12}), \mathbf{v}_{12}\rangle}{\langle \mathbf{v}_{12}, \mathbf{v}_{12}\rangle}\right| \sin\left(\frac{\pi}{r}\right) < \frac{1}{2}.$$

Squaring both sides we obtain

$$\left|\frac{\langle I_3(\mathbf{v}_{12}), \mathbf{v}_{12}\rangle}{\langle \mathbf{v}_{12}, \mathbf{v}_{12}\rangle}\right|^2 (3 - \mathrm{tr}(I_{12})) < 1$$

which is equivalent to

$$\left(\rho\sigma\tau + \overline{\rho\sigma\tau} - 2|\sigma|^2 - 2|\tau|^2 - |\rho|^2 + 4\right)^2 < 4 - |\rho|^2$$

as required. The other inequalities arise from identical arguments. □

Corollary 2.5.5 *We can rewrite these inequalities respectively as*

$$(2\mathrm{Re}\,(\mathrm{tr}\,(I_{123})) + \mathrm{tr}\,(I_{12}) - 1)^2 < 3 - \mathrm{tr}\,(I_{12}),$$
$$(2\mathrm{Re}\,(\mathrm{tr}\,(I_{123})) + \mathrm{tr}\,(I_{23}) - 1)^2 < 3 - \mathrm{tr}\,(I_{23}),$$
$$(2\mathrm{Re}\,(\mathrm{tr}\,(I_{123})) + \mathrm{tr}\,(I_{31}) - 1)^2 < 3 - \mathrm{tr}\,(I_{31}).$$

If a group $\Gamma(p, q, r; n)$ with $p > 7$ satisfies any of these inequalities the group is non-discrete.

Remark: These inequalities are the best possible, in the sense that there are discrete groups where we get equality, for example $\Gamma(18, 18, 18; 18)$ and $\Gamma(7, 7, 14; 4)$.

Theorem 2.5.6 *Let $p \le q \le r < \infty$. If $p > 31$ then the group $\Gamma(p, q, r; n)$ is not discrete.*

The structure for this proof is as follows, we first prove a technical inequality (lemma 2.5.8). We then use this lemma to prove lemma 2.5.10 which states that if $\Gamma(p, q, r; N)$ satisfies one of the Jørgensen inequalities (2.10), (2.11) or (2.12), then so will $\Gamma(p, q, r; n)$ for any $n \le N$. Finally, we show that if $p > 31$, the group $\Gamma(p, q, r; \infty)$ satisfies one of the Jørgensen inequalities for any q and r. Then, by lemma 2.5.10, $\Gamma(p, q, r; n)$ is non-discrete for all q, r and n. For the rest of this section we assume $7 \le p$.

Lemma 2.5.7 *If $7 \le p \le q \le r < \infty$, then $2\cos(\pi/7) \le |\sigma| \le |\tau| \le |\rho| < 2$.*

Lemma 2.5.8 *Let ρ, σ, τ be defined in terms of a 4-tuple $\Gamma(p, q, r; n)$ as in 2.3. Then*

$$(-2\cos(2\pi/n) + 2 + |\sigma\tau|^2 - 2|\tau|^2 - 2|\sigma|^2) \le 0$$

Proof: In section 1.5 it was shown that for a group where I_{1323} is regular elliptic of order n, we have the following equality.

$$\mathrm{Re}\,(\rho\sigma\tau) = -\frac{2\cos(2\pi/n) + 2 - |\rho|^2 - |\sigma\tau|^2}{2}$$

Then we can rewrite the inequality $-1 \leq \mathrm{Re}\,(\rho\sigma\tau)/|\rho\sigma\tau| \leq 1$ as

$$-1 \leq \frac{2\cos(2\pi/n) + 2 - |\rho|^2 - |\sigma\tau|^2}{2|\rho\sigma\tau|} \leq 1$$

Rearranging this gives us

$$(-2\cos(2\pi/n) + 2 + |\sigma\tau|^2 - 2|\tau|^2 - 2|\sigma|^2) \leq 2|\rho\sigma\tau| - |\rho|^2 - 2|\tau|^2 - 2|\sigma|^2 + 4 \qquad (2.13)$$

We can rearrange the right hand side to get

$$(-2\cos(2\pi/n) + 2 + |\sigma\tau|^2 - 2|\tau|^2 - 2|\sigma|^2) \leq -2(|\tau| - |\sigma|)^2 - 2|\sigma||\tau|(2 - |\rho|) - |\rho|^2 + 4 \qquad (2.14)$$

Since $(|\tau| - |\sigma|)^2$ is always non-negative, we have

$$(-2\cos(2\pi/n) + 2 + |\sigma\tau|^2 - 2|\tau|^2 - 2|\sigma|^2) \leq -2|\sigma||\tau|(2 - |\rho|) - |\rho|^2 + 4 \qquad (2.15)$$

The right hand side of this inequality is a quadratic in $|\rho|$

$$-|\rho|^2 + 2|\sigma\tau||\rho| + 4(1 - |\sigma\tau|).$$

This quadratic has roots 2 and $2|\sigma\tau| - 2$. Since $p \geq 7$, we have $|\sigma\tau| \geq 4\cos^2(\pi/7)$ and $(2|\sigma\tau| - 2) \geq 2$. So on the interval $0 \leq |\rho| \leq 2$ the quadratic is always non-positive. Since these are the only values that $|\rho|$ can take, the right hand side of (2.15) is always non-positive, which proves the lemma. □

Corollary 2.5.9 *Let ρ, σ, τ be defined in terms of a 4-tuple $(p, q, r; n)$ as in section 2. Then we have the following inequalities*

$$(-2\cos(2\pi/n) + 2 + |\sigma\tau|^2 - 2|\tau|^2 - |\sigma|^2 - |\rho|^2) \leq 0,$$
$$(-2\cos(2\pi/n) + 2 + |\sigma\tau|^2 - |\tau|^2 - 2|\sigma|^2 - |\rho|^2) \leq 0.$$

Proof: For the first inequality notice that

$$(-2\cos(2\pi/n) + 2 + |\sigma\tau|^2 - 2|\tau|^2 - |\sigma|^2 - |\rho|^2) - (-2\cos(2\pi/n) + 2 + |\sigma\tau|^2 - 2|\tau|^2 - 2|\sigma|^2)$$
$$= |\sigma|^2 - |\rho|^2.$$

By lemma 2.5.7 $|\sigma|^2 - |\rho|^2 \leq 0$, so using lemma 2.5.8,

$$(-2\cos(2\pi/n) + 2 + |\sigma\tau|^2 - 2|\tau|^2 - |\sigma|^2 - |\rho|^2)$$
$$\leq (-2\cos(2\pi/n) + 2 + |\sigma\tau|^2 - 2|\tau|^2 - 2|\sigma|^2) \leq 0$$

as required. The second inequality follows by essentially the same argument but using $|\tau|^2 - |\rho|^2 \leq 0$. □

Lemma 2.5.10 *If the group $\Gamma(p,q,r;N)$ satisfies any of the Jørgensen inequalities (2.10), (2.11) and (2.12) for some $N \in \mathbb{N} \cup \{\infty\}$, then $\Gamma(p,q,r;n)$ will also satisfy them for $n < N$.*

Proof: First recall that $\rho\sigma\tau + \overline{\rho\sigma\tau} = -2\cos(2\pi/n) - 2 + |\rho|^2 + |\sigma\tau|^2$. Substituting this into (2.10) gives

$$(-2\cos(2\pi/n) + 2 + |\sigma\tau|^2 - 2|\tau|^2 - 2|\sigma|^2)^2 < 4 - |\rho|^2. \tag{2.16}$$

For $n < N$, $2\cos(2\pi/n) < 2\cos(2\pi/N)$, so we have following inequality

$$0 \geq (-2\cos(2\pi/n) + 2 + |\sigma\tau|^2 - 2|\tau|^2 - 2|\sigma|^2) > (-2\cos(2\pi/N) + 2 + |\sigma\tau|^2 - 2|\tau|^2 - 2|\sigma|^2)$$

The less than zero inequality comes from lemma 2.5.8. Squaring both sides and combining with (2.16) gives

$$\begin{aligned}&(-2\cos(2\pi/n) + 2 + |\sigma\tau|^2 - 2|\tau|^2 - 2|\sigma|^2)^2\\ <&(-2\cos(2\pi/N) + 2 + |\sigma\tau|^2 - 2|\tau|^2 - 2|\sigma|^2)^2 < 4 - |\rho|^2.\end{aligned}$$

Therefore, if $\Gamma(p,q,r;N)$ satisfies (2.10), then so does $\Gamma(p,q,r;n)$ for all $n < N$. For (2.11) and (2.12) we can use the inequalities from corollary 2.5.9. □

Proof:[of proposition 2.5.6] Using lemma 2.5.10, we only need to find conditions on $\Gamma(p,q,r;\infty)$ groups. We know that if a $\Gamma(p,q,r;\infty)$ group satisfies inequalities (2.10), (2.11) and (2.12), then the group is non-discrete and then the lemma tells us that $\Gamma(p,q,r;n)$ also satisfy the inequalities and are also non-discrete. So let $n = \infty$. Then $\rho\sigma\tau + \overline{\rho\sigma\tau} = -4 + |\rho|^2 + |\sigma\tau|^2$. Substituting this into inequality (2.11) gives

$$(|\sigma\tau|^2 - |\rho|^2 - |\sigma|^2 - 2|\tau|^2)^2 < 4 - |\sigma|^2. \tag{2.17}$$

By lemma 2.5.8 the term inside the brackets on the left hand side is always negative and bounded from below by $|\sigma|^4 - 3|\sigma|^2 - 4$, to see this we use lemma 2.5.7 to obtain the following inequality (in particular we use the facts $(|\sigma|^2-2) > 0$ and $|\rho|^2 \leq 4$).

$$\begin{aligned}|\sigma\tau|^2-|\rho|^2-|\sigma|^2-2|\tau|^2 &= (|\sigma|^2-2)|\tau|^2-|\sigma|^2-|\rho|^2 \\ &\geq (|\sigma|^2-2)|\sigma|^2-|\sigma|^2-4 \\ &= |\sigma|^4-3|\sigma|^2-4.\end{aligned}$$

By lemma 2.5.8, $|\sigma\tau|^2-|\rho|^2-|\sigma|^2-2|\tau|^2$ is negative so squaring both sides will give,

$$(|\sigma|^4-3|\sigma|^2-4)^2 \geq (|\sigma\tau|^2-|\rho|^2-|\sigma|^2-2|\tau|^2)^2. \tag{2.18}$$

Combining (2.17) with (2.18), it is clear that if $|\sigma|$ satisfies

$$(|\sigma|^4-3|\sigma|^2-4)^2 < 4 - |\sigma|^2. \tag{2.19}$$

then then $|\sigma|$, $|\tau|$ and $|\rho|$ will satisfy (2.17) for any permitted $|\tau|$ and $|\rho|$, so the corresponding group $\Gamma(p, q, r; \infty)$ group is non-discrete. Then using lemma 2.5.10, all $\Gamma(p, q, r; n)$ will also be non-discrete. Expanding out the brackets and collecting terms in (2.19) gives

$$(|\sigma|^2-4)(|\sigma|^6-2|\sigma|^4-7|\sigma|^2-3) > 0. \tag{2.20}$$

By hypothesis, $\mathrm{ord}(I_{23}) \geq 7$, so $4\cos^2(\pi/7) \leq |\sigma|^2 \leq 4$. As a polynomial in $|\sigma|^2$, the left hand side of (2.20) has two roots in the interval $[4\cos^2(2\pi/7), 4]$, namely $3.9593\ldots$ and 4. When $|\sigma|^2$ lies between these roots the polynomial is negative, so (2.20) is not satisfied and the group is not discrete. So any $\Gamma(p, q, r; \infty)$ group with $3.9593\ldots \leq |\sigma|^2 \leq 4$, fails a Jørgensen discreteness test. Since $|\sigma| = 2\cos(\pi/p)$, and $4\cos^2(\pi/31) < 3.9593\ldots < 4\cos^2(\pi/32)$, then for all $p > 31$, the group $\Gamma(p, q, r; \infty)$ satisfies the inequality (2.11) and is not discrete. Then applying lemma 2.5.10, it follows for $p > 31$, $\Gamma(p, q, r; n)$ will satisfy (2.11) for all q, r, n. Therefore $\Gamma(p, q, r; n)$ is not discrete if $p > 31$. □

Conjecture 2.5.11 *Computer calculations (similar to those from the computer programme in section 2.6) strongly suggest that for $p > 22$ the group $\Gamma(p, q, r; n)$ will always fail at least one of the Jørgensen discreteness tests described in corollary 2.5.4. The largest known value of p for which there is a discrete $\Gamma(p, q, r; n)$ group* *is 18 (the group is $\Gamma(18,18,18;18)$). We conjecture that if $p > 18$ then $\Gamma(p,q,r;n)$ is not discrete.*

Remark: It is unlikely that there are similar bounds on q or r. It was conjectured in [31] that $\Gamma(4,4,4;n)$ is discrete for all $n > 4$. This group is identified with $\Gamma(4,4,n;4)$ and $\Gamma(4,n,n;4)$ via the ι maps described in 2.4. If the conjecture is correct, we can always find a discrete group $\Gamma(p,q,r;n)$ with arbitrarily large values for q or r. In [18] it is shown, using Shimizu's lemma, that for $p > 28$, the group $\Gamma(p,p,\infty;n)$ is non-discrete.

Proposition 2.5.12 *We can modify lemma 2.5.10 to obtain the following results. For any permitted $q \leq r < \infty$ and $n < \infty$, the group $\Gamma(p,q,r;n)$ is not discrete for $n < n_0(p)$ shown in the table below*

p	$n_0(p)$		p	$n_0(p)$
31	240		30	73
29	51		28	41
27	34		26	30
25	26		24	23
23	21		22	19
21	17		20	16
19	14		18	13
17	12		16	11
15	10		14	9
13	8		12	8
11	7		10	6
9	5		8	5

This result tells us that for the groups close to a deformed (∞,∞,∞) group, the parameter t corresponding to an extra discrete group outside the critical interval must be very close to the end point of the critical interval, t_1.

Proof: We will only prove the result for $\Gamma(31,q,r;n)$, all the other cases are identical. Lemma 2.5.10 tells us that if $\Gamma(p,q,r;N)$ satisfies one of the Jørgensen inequalities, then so does $\Gamma(p,q,r;n)$ for all $n \le N$. For a fixed value of p, we find the largest n such that $\Gamma(p,q,r;n)$ satisfies one of the Jørgensen inequalities for all q, r (recall if p,q,r,n satisfy any of the inequalities then $\Gamma(p,q,r;n)$ is non-discrete).

By 2.6,

$$(\rho\sigma\tau + \overline{\rho\sigma\tau}) = -2\cos(2\pi/n) - 2 + |\rho|^2 + |\sigma\tau|^2.$$

Substituting this into 2.11 gives

$$\left(-2\cos(2\pi/n) + 2 + |\sigma\tau|^2 - |\sigma|^2 - 2|\tau|^2 - |\rho|^2\right)^2 < 4 - |\sigma|^2$$

By 2.5.8 the term inside the brackets on the left hand side is non-positive. In the proof of 2.5.6, we used the following fact

$$|\sigma\tau|^2 - |\rho|^2 - |\sigma|^2 - 2|\tau|^2 \ge |\sigma|^4 - 3|\sigma|^2 - 4.$$

Combining these we get the following inequality for all τ, ρ and n (equivalently all q, r and n),

$$\left(-2\cos(2\pi/n) - 2 + |\sigma|^4 - 3|\sigma|^2\right) \le \left(-2\cos(2\pi/n) + 2 + |\sigma\tau|^2 - |\sigma|^2 - 2|\tau|^2 - |\rho|^2\right) \le 0$$

Squaring this inequality gives

$$\left(-2\cos(2\pi/n) + 2 + |\sigma\tau|^2 - |\sigma|^2 - 2|\tau|^2 - |\rho|^2\right) \le \left(-2\cos(2\pi/n) - 2 + |\sigma|^4 - 3|\sigma|^2\right)^2$$

If we fix p (and hence $|\sigma|$) it is a simple matter to find the largest n such that

$$\left(-2\cos(2\pi/n) - 2 + |\sigma|^4 - 3|\sigma|^2\right)^2 \le 4 - |\sigma|^2 \tag{2.21}$$

is satisfied. For example let $p = 31$ then $|\sigma| = 2\cos(\pi/31) = 1.989738\ldots$ and (2.21) becomes

$$(-2\cos(2\pi/n) + 1.796975506\ldots)^2 \le 0.0409401175\ldots$$

This is satisfied (and the group is non-discrete) when,

$$1.59463882\ldots \le 2\cos(2\pi/n) \le 1.99931215\ldots,$$

which lead to

$$9 < n < 240.$$

The groups with $n \leq 9$ are degenerate, so we ignore them. This gives the required result. $\square$

Corollary 2.5.13 *We have the following non-discreteness results.*

- *The group* $\Gamma(31, q, r; n)$ *is not discrete if* $q \leq \min\{r, n\} < 240$.
- *The group* $\Gamma(30, q, r; n)$ *is not discrete if* $q \leq \min\{r, n\} < 73$.
- *The group* $\Gamma(29, q, r; n)$ *is not discrete if* $q \leq \min\{r, n\} < 51$.
- *The group* $\Gamma(28, q, r; n)$ *is not discrete if* $q \leq \min\{r, n\} < 41$.
- *The group* $\Gamma(27, q, r; n)$ *is not discrete if* $q \leq \min\{r, n\} < 34$.
- *The group* $\Gamma(26, q, r; n)$ *is not discrete if* $q \leq \min\{r, n\} < 30$.
- *The group* $\Gamma(25, q, r; n)$ *is not discrete if* $q \leq \min\{r, n\} < 26$.

PROOF: When $q \leq \min\{r, n\}$ we can use the identification ι_3 described in section 2.4 to identify $\Gamma(p, q, r; n)$ to $\Gamma(p, q, n; r)$, then we apply proposition 2.5.12. (Note, the proofs of 2.5.6 and 2.5.12 required that $p \leq q \leq r$, so we require the restrictions $q \leq \min\{r, n\}$). $\square$

2.5.1 Non-standard deformed triangle groups

Recall a non-standard deformed triangle group $\Gamma(p, q, r; n/m)$ is a deformed (p, q, r) triangle group where $\mathrm{tr}(I_{1323}) = 1 + 2\cos(2m\pi/n)$, $m \neq 1$ and $\gcd(m, n) = 1$.

Corollary 2.5.14 *If* $p > 31$ *then a non-standard deformed triangle group* $\Gamma(p, q, r; n/m)$ *is non-discrete.*

PROOF: We can rewrite the arguments used in proof of theorem 2.5.6 in terms of the parameter t and the values t_0 and t_1 from section 2.3. In this notation lemma 2.5.10 says that for $t < t'$, if the deformers triangle group $\rho_{t'}(p, q, r)$ satisfies any of the Jørgensen inequalities, then so will $\rho_t(p, q, r)$. The proof of theorem 2.5.6 tells us that if $p > 31$, then the deformed triangle group $\rho_{t_1}(p, q, r)$ is non-discrete.

Since non-standard deformed triangle groups correspond to values of t in the interval (t_0, t_1), it follows all non-standard deformed triangle groups with $p > 31$ are non-discrete. $\square$

There is still the question of non-standard deformed triangle group with $p \leq 31$. We can use lemma 2.5.3 to rule out many of these groups. Let $\Gamma(p, q, r; n/m)$ be a non-standard deformed triangle group and set $A = I_{1323}$, $B = I_{23}$ and $z = p_{1323}$ in 2.5.3. Returning to the $\Gamma(4, 4, 4; 70/m)$ example, a quick calculation using Maple shows that in the only non-standard deformed triangle group that does not satisfy the Jørgensen inequality is $\Gamma(4, 4, 4; 70/3)$. We do not know whether this group is discrete, in fact we do not know if the standard deformed triangle group $\Gamma(4, 4, 4; 70)$ is discrete either.

2.6 Computer aided searches for discrete groups

We can often use the Jørgensen inequalities (2.10), (2.11) and (2.12) to quickly show that a specific $\Gamma(p, q, r; n)$ group is non-discrete. We can also quickly determine whether words of short length are elliptic or non-elliptic using theorem 1.2.12 and since I_{1323}, I_{2131} and I_{3212} have real trace we can use proposition 2.3.4 to determine if they have finite order. Combining these results we wrote a computer programme in C++ that searches for potential non-discrete $\Gamma(p, q, r; n)$ groups where all the short words are elliptic.

We make no claims about the accuracy of the programme, it is theoretically possible that we have missed some groups due to rounding errors. Although we have erred on the side caution with the rounding so we would expect to get additional ‘false positives’ rather than miss any groups. When checking the groups suggested by the programme we use traditional geometric and arithmetic methods to check for discreteness.

See [33], for example, for details on the C++ programming language.

```
#ifdef HAVE_CONFIG_H
#include <config.h>
#endif
```

```
#include <iostream>
#include <cstdlib>
#include <stdio.h>
#include <math.h>
#include <complex>
#include <sstream>

// operating with variables
using namespace std;
#define PI 3.14159265358979323846

int main ()
{

  cout << "Starting group.\n";
  int p;
  cout << "p=";
  cin >> p;
  int q;
  cout << "q=";
  cin >> q;
  int r;
  cout << "r=";
  cin >> r;
  int n;
  cout << "n=";
  cin >> n;

  int Q;
  cout << "Q=";
  cin >> Q;

  int c1,c2,c3,c4,cr,cs,crj,csj,ctj;
do{
  if(n == Q)
      ++r,n = 3;
```

```
  else
++n;
  if(r > Q)
      ++q,r = q;
  if(q > Q)
      ++p,q = p;
  if(r > Q)
r=q;

  long double rp; long double rq; long double rr;
  rp=2*cos (PI/p);rq=2*cos (PI/q);rr=2*cos (PI/r);

  long double t;
  t=  (PI- acos((2*cos((2*PI)/n)+2-rr*rr-rq*rq*rp*rp)/(2*rp*rq*rr)))/3 ;

  complex<long double> phi(cos(t),sin(t));

  long double K;long double L;long double M;
  K=-1+rr*rr+rp*rp*rq*rq-2*rp*rq*rr*cos(3*t);
  L=-1+rq*rq+rr*rr*rp*rp-2*rp*rq*rr*cos(3*t);
  M=-1+rp*rp+rq*rq*rr*rr-2*rp*rq*rr*cos(3*t);

  long double x;
  x=K*K*K*K-8*K*K*K+18*K*K-27;
if (x < 0.001)
  c1=1; // is 1323 elliptic

  x=L*L*L*L-8*L*L*L+18*L*L-27;
if (x < 0.001)
  c2=1;  //is 2131 elliptic

  x=M*M*M*M-8*M*M*M+18*M*M-27;
if (x < 0.001)
  c3=1; //is 3212 elliptic

  long double R;long double S;
  complex <long double> T;
```

```
if (c1 == 1)
  {R = 2*PI/(acos(0.5*(L-1)));}
else
  {R=1000;}
if (c3 == 1)
 {S = 2*PI/(acos(0.5*(M-1)));}
else
  {S=1000;}
  T=3-(rr*rr+rp*rp+rq*rq)+rp*rq*rr*phi*phi*phi;
  x=abs(T*T*T*T)-8*real(T*T*T)+18*abs(T*T)-27;
if (x < 0.001)
  c4=1;

if (r>6 && (2*rp*rq*rr*real(phi*phi*phi)-2*rp*rp-2*rq*rq-rr*rr+4)*
(2*rp*rq*rr*real(phi*phi*phi)-2*rp*rp-2*rq*rq-rr*rr+4) < (4-rr*rr)-0.001 )
  crj=1; // first jorg inequailty
if (p>6 && (2*rp*rq*rr*real(phi*phi*phi)-rp*rp-2*rq*rq-2*rr*rr+4)*
(2*rp*rq*rr*real(phi*phi*phi)-rp*rp-2*rq*rq-2*rr*rr+4) < (4-rp*rp)-0.001 )
  csj=1; // second jorg inequailty
if (q>6 && (2*rp*rq*rr*real(phi*phi*phi)-2*rp*rp-rq*rq-2*rr*rr+4)*
(2*rp*rq*rr*real(phi*phi*phi)-2*rp*rp-rq*rq-2*rr*rr+4) < (4-rq*rq)-0.001 )
  ctj=1; // third jorg inequailty

if(c1+c2+c3+c4 == 4 && // All Elliptic
   (abs((R)-floor(R+0.5))<0.0001) && // 2131 has finite order
   (abs((S)-floor(S+0.5))<0.0001) && // 3212 has finite order
   (rp*rq*rr*real(phi*phi*phi)-rp*rp-rq*rq-rr*rr)<-4-0.0001 && //non-degenerate
   (p!=q || q!=r || r!=p) && // not (p,p,p;n)
   crj == 0 && csj==0 && ctj==0 //Jorgensen
  )
  cout << "(" << p << "," << q << "," << r << ";"
  << n << "," << R << "," << S <<")" <<  "\n";
  c1=0;c2=0;c3=0;c4=0;crj=0;csj=0;ctj=0;

}while (p<31);
 return 0;
}
```

2.6.1 Some results from the programme

The programme does a brute force search on all 4-tuples with $p < 31$ and $q, r, n < \mathrm{Q}$, for some arbitrary value Q. It discards any 4-tuple (p, q, r, n) where I_{1323}, I_{2131}, I_{3212} and I_{123} are not all elliptic words in the group $\Gamma(p, q, r; n)$ (using Goldman's trace formula). Then it checks that I_{2131}, I_{3212} are finite order, this can easily be done since these words have real trace (lemma 2.3.4). Then it checks the group is not degenerate. Then it discards all groups of the form $\Gamma(p, p, p; n)$ since they were completely analysed by Parker in [21]. Finally it checks that the group does not satisfy any of the Jørgensen non-discreteness inequalities (2.10), (2.11) and (2.12). In the Jørgensen test we required that the relevant regular elliptic word had order at least 7. When the order is less than 7 the Jørgensen inequality is never satisfied, so is does not affect the functioning of the programme, other than making it slightly inefficient.

For initial values $p = 3$, $q = 3$, $r = 3$, $n = 3$, $\mathrm{Q} = 2000$, the programme outputs the following $(3, 3, 4; 7, 7, 7)$, $(3, 3, 5; 5, 5, 5)$, $(3, 3, 7; 4, 4, 4)$, $(3, 4, 7; 3, 3, 7)$, $(3, 5, 5; 3, 3, 10)$, $(4, 4, 5; 4, 5, 5)$, $(4, 5, 5; 4, 4, 6)$, $(4, 7, 7; 3, 3, 14)$, $(5, 5, 6; 4, 5, 5)$, $(5, 5, 10; 3, 5, 5)$ and $(7, 7, 14; 4, 7, 7)$. The first four numbers in each 6-tuple are exactly the 4-tuple $\Gamma(p, q, r; n)$, the fifth and sixth numbers are the order of I_{2131} and I_{3212} respectively. Some of these 4-tuples correspond to different presentations for the same group, after removing repeated groups we are left with the following groups,

$$\Gamma(3, 3, 4; 7),\ \Gamma(3, 3, 5; 5),\ \Gamma(4, 4, 4; 5).$$

Note that we have replaced $\Gamma(4, 4, 5; 4)$ with $\Gamma(4, 4, 4; 5)$. In addition to these three groups, $\Gamma(5, 5, 5; 5)$ also satisfies the criteria, we missed it since it is of the form $\Gamma(p, p, p; n)$, but we pick it up from Parker's paper (along with $\Gamma(4, 4, 4; 5)$).

Conjecture 2.6.1 *A deformed triangle group is a (cocompact) lattice if and only if all words of genuine length 3 and 4 are finite order regular elliptic.*

Conjecture 2.6.2 *The only deformed triangle groups with the property described in conjecture 2.6.1 are, up to the identifications ι, $\Gamma(3, 3, 4; 7)$, $\Gamma(3, 3, 5; 5)$, $\Gamma(4, 4, 4; 5)$*

and $\Gamma(5,5,5;5)$.

Combining these conjectures we get the following

Conjecture 2.6.3 *[Main Conjecture] The only deformed triangle group lattices are* $\Gamma(3,3,4;7)$, $\Gamma(3,3,5;5)$, $\Gamma(4,4,4;5)$ *and* $\Gamma(5,5,5;5)$ *(again, up to the identifications* ι*).*

Conjecture 2.6.3 follows immediately from 2.6.1 and 2.6.2. Conjecture 2.6.2 has been verified for all values of $p,q,r,n < 2000$ using the programme. For very large values of p, q, r or n the group $\Gamma(p,q,r;n)$ tends to behave like triangle groups with ideal vertices and in these groups short words tend to be loxodromic, so conjecture 2.6.2 seems reasonable.

It is less obvious why we should believe conjecture 2.6.1. In [30] Schwartz described a construction of a fundamental domain for the group $\Gamma(4,4,4;7)$ which can, in principle, be applied to any $\Gamma(p,q,r;n)$ group. If I_{123}, I_{1323}, I_{2131} or I_{3212} are non-elliptic, this construction necessarily leads to a fundamental domain with infinite volume (or at the very least cusps, hence the caveat about the lattice being cocompact). Schwartz's construction will not produce a fundamental domain for all $\Gamma(p,q,r;n)$ groups, but it suggests that if a group has short, non-elliptic words, then the group is probably not a lattice. If we have an infinite order elliptic word then the group is obviously non-discrete and not a lattice.

The groups $\Gamma(5,5,5;5)$ and $\Gamma(4,4,4;5)$ have already been studied (see [22] and [6]) and are known to be lattices; $\Gamma(3,3,4;7)$ and $\Gamma(3,3,5;5)$ are new and we analyse them in the next chapter.

In the programme, if we replace the following line of code

```
c1+c2+c3+c4 == 4 && // All Elliptic
```

with

```
c1+c2+c3 == 3 && // All Elliptic
```

then we drop the requirement that I_{123} is elliptic. The programme then outputs the following deformed triangle groups for $p,q,r,n < 1000$,

$\Gamma(3,3,r;n)$, $\Gamma(3,4,4;5)$, $\Gamma(3,5,5;3)$, $\Gamma(4,4,r;4)$, $\Gamma(4,4,5;6)$, $\Gamma(4,4,6;5)$, $\Gamma(4,4,10;3)$, $\Gamma(4,5,5;4)$, $\Gamma(4,5,5;5)$, $\Gamma(4,5,10;3)$, $\Gamma(4,7,7;3)$, $\Gamma(5,5,6;4)$, $\Gamma(5,5,6;6)$, $\Gamma(5,5,10;3)$, $\Gamma(5,5,10;4)$, $\Gamma(5,5,15;4)$, $\Gamma(6,7,7;7)$, $\Gamma(6,8,8;5)$, $\Gamma(7,7,14;4)$ and $\Gamma(9,9,18;6)$.

These groups are good candidates for interesting discrete groups; all words of the form I_{ijkj} are finite order elliptic and they do not satisfy any of the Jørgensen non-discreteness inequalities. As before the programme ignores groups of the form $\Gamma(p,p,p;n)$. Some of these groups we already know about, for example $\Gamma(5,5,6;4)$ is an alternative presentation for $\Gamma(4,4,4;5)$, which is known to be discrete. After removing repeated groups and putting the remaining groups into nicer forms we are left with the following list,

$\Gamma(3,3,r;n)$, $\Gamma(3,4,4;5)$, $\Gamma(4,4,4;n)$, $\Gamma(4,4,5;6)$, $\Gamma(4,5,5;5)$, $\Gamma(5,5,6;6)$, $\Gamma(5,5,15;4)$, $\Gamma(6,7,7;7)$, $\Gamma(6,8,8;5)$ and $\Gamma(9,9,18;6)$.

We briefly discuss whether or not these some of these groups are discrete.

$\Gamma(4,4,4;n)$

Proposition 2.6.4 *The group* $\Gamma(4,4,4;n)$ *is discrete for* $n = 5, 6, 7, 8, 10, 12, 18$.

PROOF: In corollary 1.4 of [31] it is shown that $\Gamma(4,4,4;n)$ is arithmetic, and therefore discrete, for $n = 5, 6, 7, 8, 10, 12, 18$. □

Corollary 2.6.5 *Using the identifications* ι, $\Gamma(4,4,n;4)$ *and* $\Gamma(4,n,n;4)$ *are discrete for* $n = 5,6,7,8,10,12,18$.

Conjecture 2.6.6 *The deformed triangle group* $\Gamma(4,4,4;n)$ *is discrete for all* n.

$\Gamma(3,3,4;n)$

Lemma 2.6.7 *Let* $g \in \Gamma(3,3,4;n)$. *Then* $2\mathrm{Re}\,(\mathrm{tr}(\gamma))$ *and* $|\mathrm{tr}(g)|^2$ *both belong to the ring* $\mathbb{Z}[2\cos(2\pi/n)]$.

PROOF: From corollary 18 of [26], we know the trace of any element of in $\Gamma(3,3,4;n)$

can be written as an integer polynomial in

$$|\rho|^2,\ |\sigma|^2,\ |\tau|^2,\ \rho\sigma\tau,\ \overline{\rho\sigma\tau}.$$

For a $\Gamma(3,3,4;n)$ group $|\rho|^2$, $|\sigma|^2$, $|\tau|^2$ are equal to 2, 1, 1 respectively and $2\mathrm{Re}\,(\rho\sigma\tau) = \rho\sigma\tau + \overline{\rho\sigma\tau} = 1 - 2\cos(2\pi/n)$. Therefore $2\mathrm{Re}\,(\mathrm{tr}(g))$ and $|\mathrm{tr}(g)|^2$ lie in the ring $\mathbb{Z}[2\cos(2\pi/n)]$ for any $g \in \Gamma(3,3,4;n)$. □

Theorem 2.6.8 *The group $\Gamma(3,3,4;n)$ is discrete for $n =$ 7, 8, 9, 10, 11, 12, 14, 16, 18, 24, 30.*

Proof: We use the arithmeticity criterion from chapter 1 (theorem 1.2.34). In the standard presentation the Hermitian form H for the group has determinant

$$\rho\sigma\tau + \overline{\rho\sigma\tau} - 2(|\rho|^2+|\sigma|^2+|\tau|^2) + 8$$

For $(3,3,4;n)$ this becomes

$$1 - 2\cos(2\pi/n).$$

For H to have signature (2,1), the determinate must be negative.

First observe that every element in $\mathbb{Z}[2\cos(2\pi/n)]$ is an algebraic integer in $\mathbb{Q}[2\cos(2\pi/n)]$. Let $c_0 = 2\cos(2\pi/n)$ and $c_1, c_2 \ldots, c_p$ be the non-trivial Galois conjugates of c. Let H_i be the Hermitian form obtained by applying the Galois conjugation $c_0 \mapsto c_i$ to H and $\mathrm{SU}(H_i, \mathbb{Z}[c_i])$ be the special unitary group with respect to this new Hermitian form. If H_i has positive determinant it has signature (3,0) and $\mathrm{SU}(H_i, \mathbb{Z}[c_i])$ is compact.

Let $x_0 \in \mathbb{Z}[2\cos(2\pi/n)]$ consider the map $\mathbb{Z}[2\cos(2\pi/n)] \to \mathbb{C}^{p+1}$ defined by $x_0 \mapsto (x_0, x_1, \ldots, x_p)$, where x_i are the Galois conjugates of x_0. The image of this map is a discrete set in $\mathbb{C}^{p+1}$. Hence

$$\mathrm{SU}(H_0, \mathbb{Z}[c_0]) \times \mathrm{SU}(H_1, \mathbb{Z}[c_1]) \times \ldots \times \mathrm{SU}(H_p, \mathbb{Z}[c_p])$$

is also discrete. If $\mathrm{SU}(H_i, \mathbb{Z}[2\cos(c_i)])$ is compact for all $i > 0$, then the image of the projection onto the first coordinate is also discrete, i.e. $\mathrm{SU}(H_0, \mathbb{Z}[2\cos(c_0)])$ is discrete

In this situation, the condition that $\mathrm{SU}(H_i, \mathbb{Z}[c_i])$ is compact for all $i > 0$, is equivalent to H_i having positive determinant. $\mathrm{Det}(H_i) = 1 - c_i$, so we need to find values of $n \geq 7$, such that $c_0 = 1 - 2\cos(2\pi/n)$ is positive for all non-trivial Galois conjugates of $2\cos(2\pi/n)$. The values $n = 7, 8, 9, 10, 11, 12, 14, 16, 18, 24, 30$ satisfy this condition (we believe these to be the only such values). □

Corollary 2.6.9 *Using the identifications* ι, $\Gamma(3, 3, n; 4)$, $\Gamma(3, 4, n; 3)$ *and* $\Gamma(4, n, n; 3)$ *are discrete for* $n =$ *7, 8, 9, 10, 11, 12, 14, 16, 18, 24, 30.*

Remark: Interestingly, these values for n also appear in [35], in which Takeuchi lists all arithmetic Fuchsian triangle groups. In particular a compact $(2, 3, n)$ hyperbolic triangle group is arithmetic if and only if $n =$7, 8, 9, 10, 11, 12, 14, 16, 18, 24, 30. Lemma 2.6.11 highlights another link between $(2, 3, n)$ Fuchsian triangle groups and $(3, 3, 4; n)$ deformed triangle groups.

Lemma 2.6.10 *[Lemma 3.2 of [31]] Let* H *be a group generated by complex reflections* A_1, A_2 *and* A_3 *such that* A_{ij} *has finite order* p_{ij} *for all* $i \neq j$*. If* $p_{12} = 2$*, then* H *preserves an* $\mathbb{R}$*-plane* $\mathcal{R}$ *and* $H|_{\mathcal{R}}$ *is the* (p_{12}, p_{23}, p_{31}) *triangle group.*

Lemma 2.6.11 *A* $\Gamma(3, 3, 4; n)$ *group contains a subgroup that stabilises an* $\mathbb{R}$*-plane,* $\mathcal{R}$*, and acts as the* $\mathbb{R}$*-Fuchsian* $(2, 3, n)$*-reflection group on* $\mathcal{R}$*.*

Proof: Let $H \subset \Gamma(3, 3, 4; n)$ be the subgroup generated by the following reflections

$$A_1 = I_1 I_2 I_1, \quad A_2 = I_2, \quad A_3 = I_3. \tag{2.22}$$

Then $A_1 A_2 = I_1 I_2 I_1 I_2$, has order 2, $A_2 A_3 = I_2 I_3$, has order 3 and $A_3 A_1 = I_3 I_1 I_2 I_1$, has order n. By lemma 2.6.10, H stabilises an $\mathbb{R}$-plane, $\mathcal{R}$, and $H|_{\mathcal{R}}$ acts as the $(2, 3, n)$-reflection group. □

$\Gamma(3, 3, 6; n)$

Proposition 2.6.12 *The deformed triangle group* $\Gamma(3, 3, 6; n)$ *is discrete for* $n =$*5, 6, 7, 8, 10, 12, 18.*

PROOF: This follows from an essentially identical argument to the $\Gamma(3,3,4;n)$ case above. □

Corollary 2.6.13 *Using the identifications ι, $\Gamma(3,3,n;6)$ and $\Gamma(3,6,n;3)$ are discrete for n =6, 7, 8, 10, 12, 18. When $n=5$ we need to reorder the generators to ensure that $p \leq q \leq r$, and we see that $\Gamma(3,3,5;6)$, and $\Gamma(3,5,6;3)$ are discrete.*

$\Gamma(3,3,p;n)$

Lemma 2.6.14 *In a $\Gamma(3,3,r;n)$ group, all words of the form I_{ijkj} have order n.*

PROOF: By definition $\mathrm{ord}(I_{1323})=n$. Since $I_{323}=I_{232}$, it follows that I_{1232} has order n. Conjugating I_{1323} and I_{1232} by I_3 and I_2 respectively we get I_{3212} and I_{2313}. So $\mathrm{ord}(I_{3212})=\mathrm{ord}(I_{2313})=n$. Using the relation $I_{313}=I_{131}$, we get the $\mathrm{ord}(I_{2131})=n$. Finally conjugating I_{2131} by I_1, gives us $\mathrm{ord}(I_{3121})=n$. □

Proposition 2.6.15 *The word I_{123} is loxodromic in all non-degenerate $\Gamma(3,3,p;n)$ groups except for $(p,n)=(4,7)$, $(5,5)$ and $(7,4)$.*

PROOF: By equation (2.3) the trace of I_{123} is

$$\begin{aligned}\mathrm{tr}(I_{123}) &= \rho\sigma\tau - |\rho|^2 - |\sigma|^2 - |\tau|^2 + 3\\ &= \rho\sigma\tau - |\rho|^2 + 1, \end{aligned} \tag{2.23}$$

where

$$|\rho|^2 = (2\cos(\pi/p))^2. \tag{2.24}$$

Define α to be,

$$\alpha := 2\mathrm{Re}\,(\rho\sigma\tau) = -4\cos^2(\pi/n) + |\rho|^2 + 1. \tag{2.25}$$

When we plug this into Goldman's trace formula we get

$$\begin{aligned} f(\mathrm{tr}(I_{123})) =& -4\alpha^3 + \left(-11 + |\rho|^4 + 10|\rho|^2\right)\alpha^2 + \left(-8|\rho|^4 + 14|\rho|^2 - 2|\rho|^6 + 8\right)\alpha \\ &+ \left(|\rho|^8 + 28|\rho|^2 + 6|\rho|^6 - 27|\rho|^4 - 16\right). \end{aligned} \tag{2.26}$$

First we show that for any value of $p > 3$ this cubic in α has one real root. We can ignore the case when $p = 3$ since all $\Gamma(3, 3, 3; n)$ groups are degenerate. Let $p = 4$ then the cubic becomes $-4\alpha^3 + 13\alpha^2 - 12\alpha - 4$, it is straightforward to check that this equation only has one real root.

Assume $p \geq 5$. We differentiate (2.26) to get

$$-12\alpha^2 + 2\left(-11 + |\rho|^4 + 10|\rho|^2\right)\alpha - 8|\rho|^4 + 14|\rho|^2 - 2|\rho|^6 + 8.$$

This is a quadratic in α with discriminant $868 - 72|\rho|^4 - 208|\rho|^2 - 16|\rho|^6 + 4|\rho|^8$. This discriminant is negative for $(2\cos(\pi/5))^2 \leq |\rho|^2 \leq 4$ (we can readily confirm this on Maple). So for $p \geq 5$ the derivative of the cubic 2.26 is never zero. It follows that the cubic (2.26) is monotonic and has only one real root.

We now show that if $p \geq 8$, this root lies in the interval $[2.6, 4]$. To show this we evaluate the cubic at the end points, $\alpha = 2.6, 4$. This leads to two polynomial in $|\rho|^2$

$$-139.864 - 41.04|\rho|^4 + 132.00|\rho|^2 + .8|\rho|^6 + |\rho|^8 = 2.6$$
$$-416 - 43|\rho|^4 + 244|\rho|^2 - 2|\rho|^6 + |\rho|^8 = 4.$$

On the interval $[(2\cos(\pi/8))^2, 4]$ the first of these polynomial is always positive and the second always negative (see figure 2.4). Therefore for any value of $|\rho|^2$ between $(2\cos(\pi/8))^2$ and 4 (or values of p between 8 and ∞) the only root of the cubic (2.26) lies in the interval $[2.6, 4]$. Recall that (2.26) is value of Goldman's trace formula applied to the word I_{123}. When the cubic is negative (respective positive) the word is elliptic (respectively loxodromic). Since the cubic has negative leading coefficient and one real root, it is positive if and only if α is less than the real root. We know the real root is greater than 2.6, so if α is less than 2.6, then I_{123} is loxodromic.

$$\begin{aligned} \alpha &< 2.6 \\ -(\cos(\pi/n))^2 + |\rho|^2 + 1 &< 2.6 \\ -(\cos(\pi/n))^2 &< 1.6 - (2\cos(\pi/8))^2 \\ \cos(\pi/n) &< 0.5\sqrt{(2\cos(\pi/8))^2 - 1.6} \\ n &> \frac{\pi}{\arccos(0.5\sqrt{(2\cos(\pi/8))^2 - 1.6})} \\ n &> 3.77635289\ldots \end{aligned}$$

So for all $n > 3$ and $p \geq 8$ the word I_{123} is loxodromic in the group $\Gamma(3, 3, p; n)$. When $n = 3$ the group will be degenerate, to see the observe we can use the map ι_3 turn the group into a $\Gamma(3, 3, 3; p)$ group which is always degenerate. We now deal

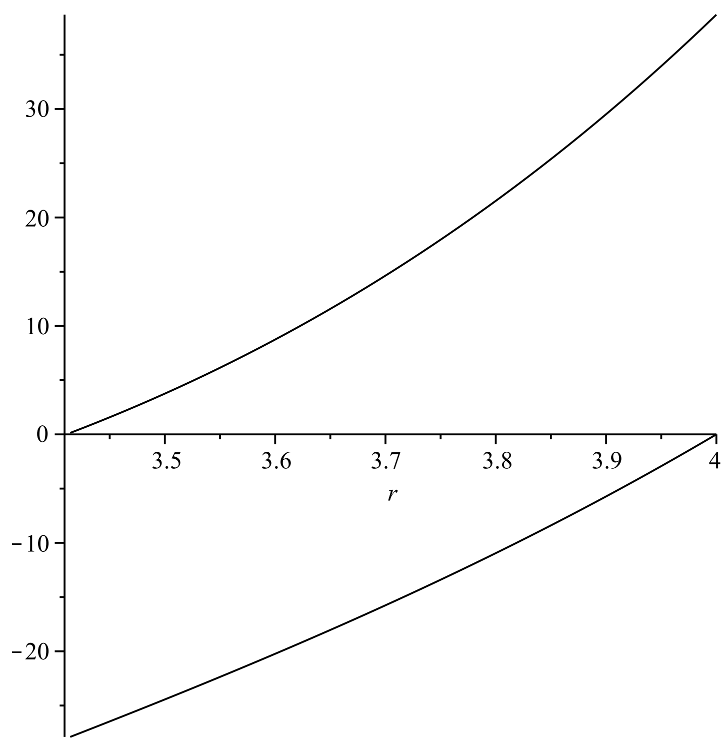

Figure 2.4: Evaluation of polynomials at $\alpha = 2.6$ and $\alpha = 4$.

with remaining cases p =4, 5, 6, 7. When $p = 4$ the real root of cubic (2.26) is $-0.2564587782\ldots$ so by the same inequality argument as above, I_{123} is loxodromic for all $n > 7.047714468\ldots$. We need to check the remaining values of n, A quick check shows that $\Gamma(3,3,4;4)$, $\Gamma(3,3,4;5)$ and $\Gamma(3,3,4;6)$ are degenerate. That leaves $\Gamma(3,3,4;7)$, in this group I_{123} is regular elliptic and has order 42. When $p=5$ the real root of (2.26) is $0.9225789220\ldots$ and we need to check all values of $n < 5.168580709\ldots$. The group $\Gamma(3,3,5;4)$ is degenerate. In the group $\Gamma(3,3,5;5)$, I_{123} is regular elliptic and has order 15. When $p=6$ the real root of (2.26) is $1.704402251\ldots$ and we need to check all values of $n<4\ldots$. There is only one group to check, namely $\Gamma(3,3,6;4)$. It is degenerate. When $p=7$ the real root of (2.26) is $2.235635507\ldots$ and we need to check all values of $n<4.014496177\ldots$. There is only one group to check, namely $\Gamma(3,3,7;4)$. In this group I_{123} is regular elliptic and has order 42. This concludes the proof. $\square$

Conjecture 2.6.16 *The deformed triangle group* $\Gamma(3,3,r;n)$ *is discrete for all* r *and* n.

$\Gamma(4,5,5;5)$

Proposition 2.6.17 *The deformed triangle group* $\Gamma(4,5,5;5)$ *is discrete.*

Let $\phi = 2\cos(\pi/5) = (1+\sqrt{5})/2$.
$\mathrm{ord}(23)=4$, $\mathrm{ord}(31)=5$, $\mathrm{ord}(12)=5$. We have

$$|\rho|^2=\phi^2,\quad |\sigma|^2=2,\quad |\tau|^2=\phi^2,\quad \mathrm{Re}\,(\rho\sigma\tau)=\phi^2.$$

From this we deduce that

$$\mathrm{Im}^2(\rho\sigma\tau)=|\rho|^2|\sigma|^2|\tau|^2-\mathrm{Re}^2(\rho\sigma\tau)=\phi^4.$$

Therefore $\rho\sigma\tau=(1+i)\phi^2$, then a solution is

$$\rho=\phi,\quad \sigma=(1+i),\quad \tau=\phi.$$

Using the standard representation we get,

$$I_1=\begin{bmatrix}1 & \phi & \phi\\ 0 & -1 & 0\\ 0 & 0 & -1\end{bmatrix},\quad I_2=\begin{bmatrix}-1 & 0 & 0\\ \phi & 1 & 1+i\\ 0 & 0 & -1\end{bmatrix},\quad I_3=\begin{bmatrix}-1 & 0 & 0\\ 0 & -1 & 0\\ \phi & 1-i & 1\end{bmatrix}. \tag{2.27}$$

preserving the Hermitian form

$$H = \begin{bmatrix} 2 & \phi & \phi \\ \phi & 2 & 1+i \\ \phi & 1-i & 2 \end{bmatrix}. \tag{2.28}$$

The Hermitian form has signature $(2,1)$. When we apply the Galois conjugation $\sqrt{5} \mapsto -\sqrt{5}$, the Hermitian form becomes signature $(3,0)$ so by arithmeticity criterion 1.2.34 the group is arithmetic and discrete.

Corollary 2.6.18 *Using the identifications ι, the following deformed triangle groups are also discrete* $\Gamma(5,5,10;4)$ *and* $\Gamma(5,10,10;5)$.

$\Gamma(5,5,15;4)$

Proposition 2.6.19 *The deformed triangle group* $\Gamma(5,5,15;4)$ *is non-discrete.*

Proof: The word I_{132323} has trace $-1+|\tau|^2-2|\sigma\tau|^2+|\rho\sigma|^2+|\tau|^2|\sigma|^4+2\mathrm{Re}\,(\rho\sigma\tau) - |\sigma|^2(2\mathrm{Re}\,(\rho\sigma\tau))$. For $\Gamma(5,5,15;4)$ this becomes

$$-1 - 2\cos(2\pi/5) - 2\cos(4\pi/5) + 2\cos(2\pi/15) = 1.82709...$$

In particular this word is elliptic with real trace, so by lemma 2.3.4, the word has finite order if there is a rational multiple of π satisfying

$$\begin{aligned} -1 - 2\cos(2\pi/5) - 2\cos(4\pi/5) + 2\cos(2\pi/15) &= 1 + 2\cos(\psi) \\ -\cos(2\pi/5) - \cos(4\pi/5) + \cos(2\pi/15) - \cos(\psi) &= 1 \end{aligned} \tag{2.29}$$

A theorem of Conway and Jones [3] lists all possible trigonometric Diophantine equations with up to four terms. We use this result to conclude that there is no rational multiple of π for ψ satisfying equation (2.29). So I_{132323} has infinite order and therefore $\Gamma(5,5,15;4)$ is non-discrete. □

$\Gamma(6,7,7;7)$

Let $\chi = 2\cos(\pi/7)$ and $\omega = (1+\sqrt{-3})/2$.
$\mathrm{ord}(23) = 6$, $\mathrm{ord}(31) = 7$, $\mathrm{ord}(12) = 7$. We have

$$|\rho|^2 = \chi^2, \quad |\sigma|^2 = 3, \quad |\tau|^2 = \chi^2, \quad \mathrm{Re}\,(\rho\sigma\tau) = 3\chi^2/2.$$

From this we deduce that

$$\mathrm{Im}^2(\rho\sigma\tau) = |\rho|^2|\sigma|^2|\tau|^2 - \mathrm{Re}^2(\rho\sigma\tau) = \sqrt{3}\chi^2/2.$$

Therefore $\rho\sigma\tau = \chi^2(1+\overline{\omega})$, then a solution is

$$\rho = \chi, \quad \sigma = (1-\overline{\omega}), \quad \tau = \chi.$$

Using the standard representation we get,

$$I_1 = \begin{bmatrix} 1 & \chi & \chi \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad I_2 = \begin{bmatrix} -1 & 0 & 0 \\ \chi & 1 & (1-\overline{\omega}) \\ 0 & 0 & -1 \end{bmatrix}, \quad I_3 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ \chi & (1-\omega) & 1 \end{bmatrix}. \tag{2.30}$$

preserving the Hermitian form

$$H = \begin{bmatrix} 2 & \chi & \chi \\ \chi & 2 & (1-\overline{\omega}) \\ \chi & (1-\omega) & 1 \end{bmatrix}. \tag{2.31}$$

Notice that χ and ω are algebraic integers. The determinant of H is $2-\chi^2 = -2\cos(2\pi/7) < 0$ and the matrix has signature $(2,1)$. For both Galois conjugates of $2\cos(\pi/7)$, $(2\cos(3\pi/7)$ and $2\cos(5\pi/7))$, H becomes positive definite and the corresponding unitary group is compact. By the same argument as 1.2.34 we conclude that the group generated by these matrices is arithmetic and therefore discrete.

Corollary 2.6.20 *Using the identifications ι, the group $\Gamma(7,7,14;6)$ is discrete.*

$\Gamma(6,8,8;5)$

Proposition 2.6.21 *The deformed triangle group $\Gamma(6,8,8;5)$ is non-discrete.*

Proof: We argue as in the $\Gamma(5,5,15;4)$ case. The word I_{132323} has trace $1 - \cos(2\pi/8) + 4\cos(2\pi/5) = 0.821854422\ldots$, so the word is elliptic and has finite order if there is a rational multiple of π satisfying.

$$1 - 2\cos(2\pi/8) + 4\cos(2\pi/5) = 1 + 2\cos(\psi)$$
$$\cos(2\pi/8) + 2\cos(2\pi/5) - \cos(\psi) = 0. \tag{2.32}$$

As before [3] tells us there are no rational solutions to (2.32), so I_{3212} has infinite order and $\Gamma(6,8,8;5)$ is non-discrete. $\square$

$\Gamma(9, 9, 18; 6)$

Proposition 2.6.22 *The deformed triangle group* $\Gamma(9, 9, 18; 6)$ *is non-discrete.*

PROOF: The word I_{132323} has trace $2\cos(2\pi/18) - 2\cos(4\pi/9) = 1.532088886\ldots$, so the word is elliptic and has finite order if there is a rational multiple of π satisfying.

$$2\cos(2\pi/18) - 2\cos(4\pi/9) = 1 + 2\cos(\psi)$$
$$\cos(2\pi/18) - \cos(4\pi/9) - \cos(\psi) = 1/2. \qquad (2.33)$$

As before [3] tells us there are no rational solutions to (2.33), so I_{3212} has infinite order and $\Gamma(9, 9, 18; 6)$ is non-discrete. □

This leaves the groups $\Gamma(3, 4, 4; 5)$, $\Gamma(4, 4, 5; 6)$ and $\Gamma(5, 5, 6; 6)$. Unfortunately we cannot determine whether or not they are discrete. These groups merit further investigation.

2.7 Non-discrete groups

Theorem 2.7.1 *For* $q, r \neq \infty$*, the following groups are non-discrete*

$$\Gamma(4, q, r; 3) \text{ where } (q, r) \neq (q, q),\ (4, 10),\ (6, r),\ (5, 10),\ (7, 14) \text{ or } (5, 15).$$
$$\Gamma(4, q, r; 4) \text{ where } (q, r) \neq (q, q),\ (4, r),\ (5, 10),\ (5, 30)\ (6, 10),\ (7, 42),\ (8, 24),$$
$$(9, 18),\ \text{or } (10, 15).$$

We don't claim the excepted groups are discrete, only that they are not necessarily non-discrete.

PROOF: The trace of I_{3212} is

$$\begin{aligned}\mathrm{tr}(I_{3212}) &= |\rho\sigma - \overline{\tau}|^2 - 1\\ &= |\rho\sigma|^2 + |\tau|^2 - 2\mathrm{Re}\,(\rho\sigma\tau) - 1\end{aligned}$$

Substituting equation (2.6) into this we get

$$\mathrm{tr}(I_{3212}) = |\rho\sigma|^2 + |\tau|^2 + 2\cos(2\pi/n) + 1 - |\rho|^2 - |\sigma\tau|^2.$$

In the group $\Gamma(4, q, r; 3)$, $|\sigma|^2 = 2$ and $2\cos(2\pi/n) = -1$, so this becomes

$$\mathrm{tr}(I_{3212}) = |\rho|^2 - |\tau|^2.$$

Since $2 \leq |\tau|^2 \leq |\rho|^2 \leq 4$, the trace is always a real number between -1 and 3, so the word I_{3212} is elliptic. Now we employ an argument from the proof of lemma 2.3.4, namely if an elliptic word has real trace then the trace is exactly $1 + 2\cos(\psi)$ for some angle ψ. Moreover the word has finite order if and only if ψ is a rational multiple of π. So I_{3212} has finite order if and only ψ is a rational multiple of π satisfying

$$\cos\left(\frac{2\pi}{r}\right) - \cos\left(\frac{2\pi}{q}\right) - \cos(\psi) = \frac{1}{2}.$$

We use Conway and Jones [3] to find all the rational solutions to this Diophantine trigonometric equation. These solutions are

- $\cos\left(\frac{2\pi}{q}\right) - \cos\left(\frac{2\pi}{q}\right) - \cos\left(\frac{2\pi}{3}\right) = \frac{1}{2}$,
- $\cos\left(\frac{2\pi}{4}\right) - \cos\left(\frac{2\pi}{6}\right) - \cos\left(\frac{2\pi}{4}\right) = \frac{1}{2}$,
- $\cos\left(\frac{2\pi}{4}\right) - \cos\left(\frac{2\pi}{10}\right) - \cos\left(\frac{2\pi}{5}\right) = \frac{1}{2}$,
- $\cos\left(\frac{2\pi}{r}\right) - \cos\left(\frac{2\pi}{3}\right) - \cos\left(\frac{2\pi}{r}\right) = \frac{1}{2}$,
- $\cos\left(\frac{2\pi}{10}\right) - \cos\left(\frac{2\pi}{5}\right) - \cos\left(\frac{2\pi}{2}\right) = \frac{1}{2}$,
- $\cos\left(\frac{2\pi}{14}\right) - \cos\left(\frac{2\pi}{7}\right) - \cos\left(\frac{4\pi}{7}\right) = \frac{1}{2}$,
- $\cos\left(\frac{2\pi}{15}\right) - \cos\left(\frac{2\pi}{5}\right) - \cos\left(\frac{7\pi}{15}\right) = \frac{1}{2}$.

These correspond to the groups $\Gamma(4, q, q; 3)$, $\Gamma(4, 4, 6; 3)$, $\Gamma(4, 4, 10; 3)$, $\Gamma(4, 6, r; 3)$ $\Gamma(4, 5, 10; 3)$, $\Gamma(4, 7, 14; 3)$ and $\Gamma(4, 5, 15; 3)$. In all other $\Gamma(4, p, r; 3)$ groups, I_{3212} is an infinite order elliptic word and therefore the group is non-discrete. We also disregard $\Gamma(4, 4, 6; 3)$ since it is degenerate.

For groups of the form $\Gamma(4, q, r; 4)$, $|\sigma|^2 = 2$ and $2\cos(2\pi/n) = 0$, so the trace of I_{3212} becomes

$$\mathrm{tr}(I_{3212}) = |\rho|^2 - |\tau|^2 + 1.$$

As before this is always real and between -1 and 3, so I_{3212} is elliptic. We set this equal to $2\cos(\psi) + 1$ to get

$$\cos\left(\frac{2\pi}{r}\right) - \cos\left(\frac{2\pi}{q}\right) - \cos(\psi) = 0.$$

Using Conway and Jones we get the following solutions to this equation. These are the only rational solutions.

- $\cos\left(\frac{2\pi}{q}\right) - \cos\left(\frac{2\pi}{q}\right) - \cos\left(\frac{\pi}{2}\right) = 0,$
- $\cos\left(\frac{2\pi}{4}\right) - \cos\left(\frac{2\pi}{r}\right) - \cos\left(\frac{(r-2)\pi}{2}\right) = 0,$
- $\cos\left(\frac{2\pi}{10}\right) - \cos\left(\frac{2\pi}{5}\right) - \cos\left(\frac{\pi}{3}\right) = 0,$
- $\cos\left(\frac{2\pi}{10}\right) - \cos\left(\frac{2\pi}{6}\right) - \cos\left(\frac{2\pi}{5}\right) = 0,$
- $\cos\left(\frac{2\pi}{30}\right) - \cos\left(\frac{2\pi}{5}\right) - \cos\left(\frac{4\pi}{15}\right) = 0,$
- $\cos\left(\frac{2\pi}{42}\right) - \cos\left(\frac{2\pi}{7}\right) - \cos\left(\frac{8\pi}{21}\right) = 0,$
- $\cos\left(\frac{2\pi}{24}\right) - \cos\left(\frac{2\pi}{8}\right) - \cos\left(\frac{5\pi}{12}\right) = 0,$
- $\cos\left(\frac{2\pi}{18}\right) - \cos\left(\frac{2\pi}{9}\right) - \cos\left(\frac{4\pi}{9}\right) = 0,$
- $\cos\left(\frac{2\pi}{15}\right) - \cos\left(\frac{2\pi}{10}\right) - \cos\left(\frac{7\pi}{15}\right) = 0.$

These correspond to the groups $\Gamma(4, q, q; 4)$, $\Gamma(4, 4, r; 4)$, $\Gamma(4, 5, 10; 4)$ $\Gamma(4, 6, 10; 4)$, $\Gamma(4, 5, 30; 4)$, $\Gamma(4, 7, 42; 4)$, $\Gamma(4, 8, 24; 4)$, $\Gamma(4, 9, 18; 4)$, $\Gamma(4, 10, 15; 4)$. All other groups of the form $\Gamma(4, q, r; 4)$ are non-discrete. □

This theorem is conceptually similar to theorem 3.2.0.13 of [36], where it is shown by a similar argument that $\Gamma(4, \infty, \infty; n)$ is non-discrete for $n \neq 3, 4$.

We can generalise this argument to more general $\Gamma(p,q,r;n)$ groups. But unless $2\cos(\pi/p)$ is an integer, the Diophantine trigonometric equation becomes too large to effectively tackle. We can, however, obtain some partial results for $\Gamma(4,q,r;n)$.

Theorem 2.7.2 *Let $\Gamma(4,q,r;n)$ be a deformed triangle group with I_{3212} elliptic. Then $\Gamma(4,q,r;n)$ is non-discrete unless it is one of the following groups.*

$$\Gamma(4,4,4;n),\ \Gamma(4,4,5;6),\ \Gamma(4,4,6;5),\ \ \Gamma(4,10,30;6),\ \Gamma(4,4,10;3),\ \Gamma(4,4,r;4),$$
$$\Gamma(4,5,10;3),\ \Gamma(4,5,10;4),\ \Gamma(4,5,15;3),\ \Gamma(4,5,30;4),\ \Gamma(4,6,6;n),\ \Gamma(4,6,10;4),$$
$$\Gamma(4,6,r;3),\ \Gamma(4,6,r;4),\ \Gamma(4,6,r;6),\ \Gamma(4,7,14;3),\ \Gamma(4,7,42;4),\ \Gamma(4,8,24;4),$$
$$\Gamma(4,9,18;4),\ \Gamma(4,10,15;4),\ \Gamma(4,q,q;n),\ \Gamma(4,q,r;q).$$

Again, we don't claim all these groups are discrete, only that they are not necessary non-discrete. Also we say nothing about groups where I_{3212} is non-elliptic.

Proof: In a $\Gamma(4,q,r;n)$ group the trace of I_{3212} is

$$\operatorname{tr}(I_{3212}) = 2\cos(2\pi/r) - 2\cos(2\pi/q) + 2\cos(2\pi/n) + 1. \tag{2.34}$$

Unlike the previous theorem, this does not always lie between -1 and 3, so we need put bounds on q, r and n to ensure the word is elliptic before we employ the same arguments. In order for the word to be elliptic we need $\cos(2\pi/r) + \cos(2\pi/n) < 1 + \cos(2\pi/q)$. We also assume $n > 4$, since all group of the form $\Gamma(4,q,r;4)$ and $\Gamma(4,q,r;3)$ were dealt with in the previous theorem. We prove the result by finding all solutions to the trigonometric Diophantine equation when $q = 4$, $q = 5$ or $q = 6$ and then showing that if $q \neq 4,5,6$ there are no solutions.

Let $q = 4$, then $4 \leq r$ and I_{3212} is elliptic when $\cos(2\pi/r)+\cos(2\pi/n) < 1$. Since we have already ruled out $n = 3$, 4, the only values of (r,n) for which I_{3212} is elliptic are $(4,n)$, $(5,5)$, $(5,6)$, $(5,7)$, $(6,5)$, $(6,6)$, $(7,5)$. Recall that we require $p \leq q \leq r$. Then setting

$$\operatorname{tr}(I_{3212}) = 2\cos(2\pi/r) + 2\cos(2\pi/n) + 1$$

equal to $2\cos\psi + 1$, we get the following equation,

$$\cos(2\pi/r) + \cos(2\pi/n) - \cos(\psi) = 0.$$

Substituting the values of (r, n) into this equation we get the following,

$$\begin{array}{lll} a) & \cos(2\pi/n) - \cos(\psi) = 0 & (4, n) \\ b) & \cos(2\pi/5) + \cos(2\pi/5) - \cos(\psi) = 0 & (5, 5) \\ c) & -\cos(2\pi/5) + \cos(\psi) = 1/2 & (5, 6) \\ d) & \cos(2\pi/5) + \cos(2\pi/7) - \cos(\psi) = 0 & (5, 7) \\ e) & -\cos(2\pi/5) + \cos(\psi) = 1/2 & (6, 5) \\ f) & \cos(2\pi/7) + \cos(2\pi/5) - \cos(\psi) = 0 & (7, 5) \end{array}$$

Notice that the equations $c)$ and $e)$ (also $d)$ and $f)$) are the same, this is related to the map ι_3 identifying the groups $\Gamma(p, q, r; n)$ and $\Gamma(p, q, n; r)$. As before we are searching for ψ a rational multiple of π that satisfies these equations. Clearly $a)$ has a solution for all n so we can't say anything about the non-discreteness of the groups $\Gamma(4, 4, 4; n)$. However we can use Conway and Jones to see that $b)$, $d)$ and $f)$ do not have solutions, so $\Gamma(4, 4, 5; 5)$, $\Gamma(4, 4, 5; 7)$ and $\Gamma(4, 4, 7; 5)$ are non-discrete. Finally, equation $c)$ does have a solution namely

$$-\cos(2\pi/5) + \cos(\pi/5) = 1/2$$

so $\Gamma(4, 4, 5; 6)$ and $\Gamma(4, 4, 6; 5)$ may be discrete.

Let $q = 5$ then $4 \leq r$ and I_{3212} is elliptic when $\cos(2\pi/r) + \cos(2\pi/n) < 1$. Since we have already ruled out $n = 3$, 4, the only values of (r, n) for which I_{3212} is elliptic are $(5, n)$, $(6, 6)$, $(6, 7)$, $(6, 8)$, $(6, 9)$, $(7, 6)$, $(7, 7)$, $(8, 6)$, $(9, 6)$, $(r, 5)$. Then setting

$$\mathrm{tr}(I_{3212}) = 2\cos(2\pi/r) - 2\cos(2\pi/5) + 2\cos(2\pi/n) + 1. \tag{2.35}$$

equal to $2\cos\psi + 1$, we get the following equation,

$$\cos(2\pi/r) - \cos(2\pi/5) + \cos(2\pi/n) - \cos\psi = 0.$$

Substituting the values of (r, n) into this equation we get the following,

$$
\begin{array}{llr}
a) & \cos(2\pi/n) - \cos\psi = 0 & (5, n) \\
b) & \cos(2\pi/5) + \cos\psi = 1 & (6, 6) \\
c) & \cos(2\pi/5) - \cos(2\pi/7) + \cos\psi = 1/2 & (6, 7) \\
d) & \cos(2\pi/5) - \cos(2\pi/8) + \cos\psi = 1/2 & (6, 8) \\
e) & \cos(2\pi/5) - \cos(2\pi/9) + \cos\psi = 1/2 & (6, 9) \\
f) & \cos(2\pi/5) - \cos(2\pi/7) + \cos\psi = 1/2 & (7, 6) \\
g) & \cos(2\pi/7) - \cos(2\pi/5) + \cos(2\pi/7) - \cos\psi = 0 & (7, 7) \\
h) & \cos(2\pi/5) - \cos(2\pi/8) + \cos\psi = 1/2 & (8, 6) \\
i) & \cos(2\pi/5) - \cos(2\pi/9) + \cos\psi = 1/2 & (9, 8) \\
j) & \cos(2\pi/q) - \cos\psi = 0 & (r, 5)
\end{array}
$$

Using [3], the only equations in this list with rational solutions are $a)$ and $j)$, so we conclude $\Gamma(4, 5, 5; n)$ and $\Gamma(4, 5, r; 5)$ may be discrete while $\Gamma(4, 5, 6; 6)$, $\Gamma(4, 5, 6; 7)$, $\Gamma(4, 5, 6; 8)$, $\Gamma(4, 5, 6; 9)$, $\Gamma(4, 5, 7; 6)$, $\Gamma(4, 5, 7; 7)$, $\Gamma(4, 5, 8; 6)$ and $\Gamma(4, 5, 9; 8)$ are non-discrete.

Let $q = 6$, note that using the map ι_3, when $q \leq r, n$ we can swap r and n without changing the group, this is reflected in the fact r and n are symmetric in the trace equation. We use this to reduce the number cases we need to test.

The word I_{3212} is elliptic when $\cos(2\pi/r) + \cos(2\pi/n) < 1.5$ for $6 \leq r$. As before we have already ruled out $n = 3$, 4, so the only values of (r, n) for which I_{3212} is elliptic are $(r, 5)$, $(r, 6)$, $(7, 7)$, $(7, 8)$, $(7, 9)$, $(7, 10)$, $(7, 11)$, $(7, 12)$, $(8, 8)$, $(8, 9)$. Then setting $\mathrm{tr}(I_{3212})$ equal to $2\cos\psi + 1$ and substituting the listed values of (r, n) into the equation, we get the following,

$$
\begin{array}{lll}
a) & \cos(2\pi/r) + \cos(2\pi/5) - \cos(\psi) = 1/2 & (r, 5) \\
b) & \cos(2\pi/r) - \cos(\psi) = 0 & (r, 6) \\
c) & \cos(2\pi/7) + \cos(2\pi/7) - \cos(\psi) = 1/2 & (7, 7) \\
d) & \cos(2\pi/7) + \cos(2\pi/8) - \cos(\psi) = 1/2 & (7, 8) \\
e) & \cos(2 - pi/7) + \cos(2\pi/9) - \cos(\psi) = 1/2 & (7, 9) \\
f) & \cos(2\pi/7) + \cos(2\pi/10) - \cos(\psi) = 1/2 & (7, 10) \\
g) & \cos(2\pi/7) + \cos(2\pi/11) - \cos(\psi) = 1/2 & (7, 11) \\
h) & \cos(2\pi/7) + \cos(2\pi/12) - \cos(\psi) = 1/2 & (7, 12) \\
g) & \cos(2\pi/8) + \cos(2\pi/8) - \cos(\psi) = 1/2 & (8, 8) \\
i) & \cos(2\pi/8) + \cos(2\pi/9) - \cos(\psi) = 1/2 & (8, 9)
\end{array}
$$

Using Conway and Jones we quickly establish that there are no solutions to equations a, $c) - i)$. Equation $b)$ has a trivial solution for all r. So the groups $\Gamma(4, 6, r; 5)$, $\Gamma(4, 6, 7; 7)$, $\Gamma(4, 6, 7; 8)$, $\Gamma(4, 6, 7; 9)$, $\Gamma(4, 6, 7; 10)$, $\Gamma(4, 6, 7; 11)$, $\Gamma(4, 6, 7; 12)$, $\Gamma(4, 6, 8; 8)$ and $\Gamma(4, 6, 8; 9)$ are non-discrete. The groups $\Gamma(4, 6, r; 6)$ (and $\Gamma(4, 6, 6; n)$) may be discrete for all r (and n).

In principle we could continue this procedure indefinitely for increasingly large values of q, however this is not necessary. Notice that when $2\cos(2\pi/q)$ is not an integer the trigonometric Diophantine equation we get when I_{3212} is elliptic is

$$\cos(2\pi/r) - \cos(2\pi/q) + \cos(2\pi/n) - \cos(\psi) = 0.$$

This equation has very few solutions, in particular if $n \neq q \neq r$ and at least one of q, r, n is not equal to 4 or 6, there are no solutions. When either $n = q$ or $r = q$ there is an obvious solution, so we can't say anything about the non-discreteness of $\Gamma(4, q, q; n)$ or $\Gamma(4, q, r; q)$.

We have already dealt with the cases $q = 4, 5, 6$ above and $n = 4$ was done in theorem 2.7.1. Since $q \leq r$ we don't have to treat the case $r = 4$ since $r = 4 \implies q = 4$. Therefore the only possible remaining discrete groups $\Gamma(4, q, r; n)$ with elliptic I_{3212}, occur when $r = 6$ or $n = 6$.

Let $r = 6$, then the trace of I_{3212} is

$$\operatorname{tr}(I_{3212}) = -2\cos(2\pi/q) + 2\cos(2\pi/n) + 2. \tag{2.36}$$

So I_{3212} is elliptic if (q,n) is one of the following $(4,5)$, $(5,6)$, $(5,7)$, $(5,8)$, $(5,9)$, $(6,n)$, we exclude the case where $4 \leq q \leq 6$, $q \neq n$ and $n = 3$ or 4 since they have either already been dealt with or. Setting the trace equal to $1 + 2\cos(\psi)$ we get the equation

$$\cos(2\pi/q) - \cos(2\pi/n) + \cos(\psi) = 1/2. \tag{2.37}$$

Substituting the values for (q,n) above we get the following equations

$$\begin{array}{lll}
a) & -\cos(2\pi/5) + \cos(\psi) = 1/2 & (4,5) \\
b) & \cos(2\pi/5) + \cos(\psi) = 1 & (5,6) \\
c) & \cos(2\pi/5) - \cos(2\pi/7) + \cos(\psi) = 1/2 & (5,7) \\
d) & \cos(2\pi/5) - \cos(2\pi/8) + \cos(\psi) = 1/2 & (5,8) \\
e) & \cos(2\pi/5) - \cos(2\pi/9) + \cos(\psi) = 1/2 & (5,9) \\
f) & -\cos(2\pi/n) + \cos(\psi) = 0 & (6,n)
\end{array}$$

The only equations with rational solutions are a and $f)$ so we conclude $\Gamma(4,4,6;5)$ and $\Gamma(4,6,6;n)$ may be discrete and $\Gamma(4,5,6;6)$, $\Gamma(4,5,6;7)$, $\Gamma(4,5,6;8)$ and $\Gamma(4,5,6;9)$ are non-discrete.

Let $n = 6$, then the trace of I_{3212} is

$$\mathrm{tr}(I_{3212}) = -2\cos(2\pi/q) + 2\cos(2\pi/r) + 2. \tag{2.38}$$

This is the same equation as in the previous case but we obtain slightly different values for pairs (q,r) reflecting the fact the we have more freedom varying q and r since neither is bounded from above by n. These pairs are $(4,5)$, $(5,6)$, $(5,7)$, $(5,8)$, $(5,9)$, $(6,r)$ and (q_0, r_0), where $6 < q_0 \leq r_0$. These lead to the equations

$$\begin{array}{lll}
a) & -\cos(2\pi/5) + \cos(\psi) = 1/2 & (4,5) \\
b) & \cos(2\pi/5) + \cos(\psi) = 1 & (5,6) \\
c) & \cos(2\pi/5) - \cos(2\pi/7) + \cos(\psi) = 1/2 & (5,7) \\
d) & \cos(2\pi/5) - \cos(2\pi/8) + \cos(\psi) = 1/2 & (5,8) \\
e) & \cos(2\pi/5) - \cos(2\pi/9) + \cos(\psi) = 1/2 & (5,9) \\
f) & -\cos(2\pi/r) + \cos(\psi) = 0 & (6,r) \\
g) & \cos(2\pi/q) - \cos(2\pi/r) + \cos(\psi) = 1/2 & (q,r)\ q > 6
\end{array}$$

Equations $a) - f)$ are exactly the same as in the previous case so a and $f)$ have rational solutions while $b)$, $c)$, $d)$, $e)$ do not. This only leaves $g)$, which by Conway and Jones has one solution $q = 10$, $r = 30$ $\psi = 4\pi/15$. So we conclude the only possible discrete groups of the form $\Gamma(4, q, r; 6)$ are $\Gamma(4, 4, 4; 5)$, $\Gamma(4, 5, 6; 6)$, $\Gamma(4, 6, r; 6)$ and $\Gamma(4, 10, 30; 6)$. □

This theorem is similar to the main result of [27], where Pratoussevitch shows that if the word I_{123} is regular elliptic in a deformed (m, m, ∞) group, then it is necessarily infinite order and the group is non-discrete. Here we show that if I_{3212} is regular elliptic in a deformed $(4, q, r)$ group then, apart from the exceptional groups listed, I_{3212} is infinite order.

In principle we could go through a similar procedure for $\Gamma(3, q, r; n)$ and $\Gamma(6, q, r; n)$ groups. If there were a list of trigonometric Diophantine equations analogous to [3] but with up to eight cosine terms rather than four, we could apply the procedure to all $\Gamma(p, q, r; n)$ groups with I_{3212} elliptic. This style of argument may yield some results for groups of the form (p, q, q) or (p, p, q) since the repeated variable simplifies the resulting equation somewhat.

We could also go through a similar procedure using the trace of I_{2131}. However this word becomes loxodromic before I_{3212}, i.e. if I_{2131} is elliptic then I_{3212} also elliptic.

2.7.1 Non-standard deformed triangle groups revisited

In the above analysis if ψ is of the form $2m\pi/l$ for $m \neq 1$ then the trace of I_{3212} is $1 + 2\cos(2m\pi/l)$ and we can rewrite the group as a non-standard deformed triangle group. We can apply the ι_2 map from section 2.4 and relabel the resulting generators to change these standard groups into non-standard groups as follows

Standard	Non − standard
$\Gamma(4,5,15;3)$	$\Gamma(3,4,5;30/7)$
$\Gamma(4,7,14;3)$	$\Gamma(3,4,7;7/2)$
$\Gamma(4,5,30;4)$	$\Gamma(4,4,5;15/2)$
$\Gamma(4,7,42;4)$	$\Gamma(4,4,7;21/4)$
$\Gamma(4,8,24;4)$	$\Gamma(4,4,8;24/5)$
$\Gamma(4,9,18;4)$	$\Gamma(4,4,9;9/2)$
$\Gamma(4,10,15;4)$	$\Gamma(4,4,10;30/7)$
$\Gamma(4,10,30;6)$	$\Gamma(4,6,10;15/2)$

We can apply the Jørgensen test from section 2.5.1 to these groups. This shows that the non-standard deformed triangle groups $\Gamma(3,4,5;30/7)$, $\Gamma(4,4,7;21/4)$, $\Gamma(4,4,8;24/5)$ and $\Gamma(4,4,10;30/7)$ are non-discrete. Consequently, the standard deformed triangle groups $\Gamma(4,5,15;3)$, $\Gamma(4,7,14;3)$, $\Gamma(4,8,24;4)$ and $\Gamma(4,10,15;4)$ are non-discrete.

Question

Are the deformed triangle groups

$\Gamma(4,7,14;3)$, $\Gamma(4,5,30;4)$, $\Gamma(4,9,18;4)$, $\Gamma(4,10,30;6)$ discrete?

equivalently,

Are the non-standard deformed triangle groups

$\Gamma(3,4,7;7/2)$, $\Gamma(4,4,5;15/2)$, $\Gamma(4,4,9;9/2)$, $\Gamma(4,6,10;15/2)$ discrete?

Chapter 3

$\Gamma(3,3,4;7)$ and $\Gamma(3,3,5;5)$

In the previous chapter, the computer programme suggested two new candidates for deformed triangle group lattices, $\Gamma(3,3,4;7)$ and $\Gamma(3,3,5;5)$. In this chapter we prove that they are lattices, give commensurability results, calculate their covolume and give presentations for the groups. Then we construct conjectural fundamental domains which agrees with the given covolumes and presentations.

3.1 Representations of $\Gamma(3,3,4;7)$ and $\Gamma(3,3,5;5)$ in $\mathbf{PU}(2,1)$

3.1.1 A representation of $\Gamma(3,3,4;7)$ in $\mathbf{PU}(2,1)$

Let $u = e^{2i\pi/7}$. To fix the orders of I_{23}, I_{31} and I_{12}, we require that

$$|\sigma|^2 = 1, \quad |\tau|^2 = 1, \quad |\rho|^2 = 2.$$

For I_{1323} to have order 7 we require that

$$|\sigma\tau - \overline{\rho}| = 2\cos(\pi/7).$$

A solution to these equation is given by

$$\sigma = u^5, \quad \tau = u^5, \quad \rho = u + u^2 + u^4.$$

Then we have the following representation for $\Gamma(3,3,4;7)$,

$$I_1 = \begin{pmatrix} 1 & u+u^2+u^4 & u^2 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad I_2 = \begin{pmatrix} -1 & 0 & 0 \\ u^6+u^5+u^3 & 1 & u^5 \\ 0 & 0 & -1 \end{pmatrix},$$

$$I_3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ u^5 & u^2 & 1 \end{pmatrix}.$$

The Hermitian form preserved by the group is

$$H = \begin{pmatrix} 2 & u+u^2+u^4 & u^2 \\ u^6+u^5+u^3 & 2 & u^5 \\ u^5 & u^2 & 2 \end{pmatrix}.$$

3.1.2 A representation of $\Gamma(3,3,5;5)$ in $\mathbf{PU}(2,1)$

Let $u = e^{2i\pi/5}$. To fix the orders of I_{23}, I_{31} and I_{12}, we require that

$$|\sigma|^2 = 1, \quad |\tau|^2 = 1, \quad |\rho|^2 = (3+\sqrt{5})/2.$$

For I_{1323} to have order 5 we require that

$$|\sigma\tau - \overline{\rho}| = 2\cos(\pi/5).$$

A solution to these equation is given by

$$\sigma = u^2, \quad \tau = u^2, \quad \rho = -1-u^4.$$

Then we have the following representation for $\Gamma(3,3,5;5)$,

$$I_1 = \begin{pmatrix} 1 & -1-u^4 & u^3 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad I_2 = \begin{pmatrix} -1 & 0 & 0 \\ -1-u & 1 & u^2 \\ 0 & 0 & -1 \end{pmatrix},$$

$$I_3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ u^2 & u^3 & 1 \end{pmatrix}.$$

The Hermitian form preserved by the group is

$$H = \begin{pmatrix} 2 & -1-u^4 & u^3 \\ -1-u & 2 & u^2 \\ u^2 & u^3 & 2 \end{pmatrix}.$$

Corollary 3.1.1 *The word I_{123} has order 42 (respectively 15) in the group $\Gamma(3,3,4;7)$ (respectively $\Gamma(3,3,5;5)$).*

Proof: This follows from analysis of eigenvalues or straightforward multiplication of the matrices using computer software e.g. Maple. □

3.2 Commensurability

In his thesis, [15], Livné discovered a number of lattices in $\mathbf{PU}(2,1)$ that are related to $\Gamma(n,n,n;n)$ triangle groups for n =5, 6, 7, 8, 9, 10, 12 and 18. In the language of Mostow [4], these lattices are $\Gamma(5,\frac{1}{2})$, $\Gamma(6,\frac{1}{3})$, $\Gamma(7,\frac{3}{14})$, $\Gamma(8,\frac{1}{8})$, $\Gamma(9,\frac{1}{18})$, $\Gamma(10,0)$, $\Gamma(12,\frac{1}{12})$ and $\Gamma(18,\frac{2}{9})$. The Livné lattice $\Gamma(n,*)$ contains the corresponding $\Gamma(n,n,n;n)$ groups as an infinite index subgroup (except for $p=5$ where the group is an index 60 subgroup). The lattices have a presentation of the form

$$\langle A_1, P, R_1 \mid A_1^2 = P^d = R_1^p = (PA_1)^3 = Id,\ R_1A_1 = A_1R_1,\ R_1PR_1 = P^2\rangle. \quad (3.1)$$

(section 5.2 of [23]). The values of d and p depend on the groups. In particular, for $\Gamma(10,0)$, $d=15$ and $p=10$. For $\Gamma(7,\frac{3}{14})$, $d=42$ and $p=7$.

Proposition 3.2.1 *The group $\Gamma(3,3,5;5)$ contains $\Gamma(10,10,10;10)$ as a subgroup and the group $\Gamma(3,3,4;7)$ contains $\Gamma(7,7,7;7)$ as a subgroup.*

Proof: In both cases the subgroup in question is generated by the reflections $A_1 = I_1$, $A_2 = I_{32123}$ and $A_3 = I_{23132}$, then using the presentations given above we check the traces of the relevant words to confirm that the following relations hold in the group: A_i^2, A_{ij}^p, A_{ijkj}^p, where $p=10$ resp. 7. There are extra relations in these groups, but determining the traces (and therefore the order) of the words

A_{ij} and A_{1323} is enough to uniquely determine the group. Therefore $\langle A_1, A_2, A_3\rangle = \Gamma(7,7,7;7)$ resp. $\Gamma(10,10,10;10)$. □

Theorem 3.2.2 *The groups* $\Gamma(3,3,4;7)$ *and* $\Gamma(7,\frac{3}{14})$ *are commensurable and the groups* $\Gamma(3,3,5;5)$ *and* $\Gamma(10,0)$ *are commensurable.*

We will give a longer proof of theorem 3.2.2 than is strictly necessary, in particular we calculate the coset decompositions and precise indices for the subgroups, rather than just demonstrating that they are finite index. This allows us to calculate the covolumes of $\Gamma(3,3,4;7)$ and $\Gamma(3,3,5;5)$ in section 3.2.15. An immediate consequence of this commensurability relation is that the groups $\Gamma(3,3,4;7)$ and $\Gamma(3,3,5;5)$ are arithmetic lattices.

We begin by showing that $\Gamma(3,3,4;7)$ and $\Gamma(7,\frac{3}{14})$ are commensurable.

Lemma 3.2.3 *Consider the subgroups*

$$G := \langle I_1, I_{321}, I_{1232123212}I_{321}I_{2123212321}\rangle \subset \Gamma(3,3,4;7)$$

and

$$G' := \langle A_1, P, R_1^2PR_1^{-2}\rangle \subset \Gamma(7,\tfrac{3}{14})$$

There is matrix C such that

- $C^{-1}A_1C = I_1$,
- $C^{-1}PC = I_{321}$,
- $C^{-1}(R_1^2PR_1^{-2})C = I_{1232123212}I_{321}I_{2123212321}$ *and*
- $C^*H_LC = H$.

In other words $\Gamma(3,3,4;7)$ *and* $C^{-1}\left(\Gamma(7,\frac{3}{14})\right)C$ *share a common subgroup namely* $\langle I_1, I_{321}, I_{1232123212}I_{321}I_{2123212321}\rangle$.

Proof: Let $u = e^{2i\pi/7}$. In [22] Parker gives the following representation for $\Gamma(7,\frac{3}{14})$.

$$R_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & u & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad A_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$P = \frac{-u^5}{2(1-u)} \begin{pmatrix} u-u^6 & -2+u+u^6 & 2-u-u^6 \\ -2u & -2 & 2u \\ -u-u^6 & -u-u^6 & 2+u-u^6 \end{pmatrix},$$

preserving the Hermitian form

$$H_L = (u-u^6) \begin{pmatrix} -1/(2-u-u^6) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/(u+u^6) \end{pmatrix}.$$

Then the required conjugation matrix is

$$C = \alpha \begin{pmatrix} 2(1-u)(u^3-u^6) & (1-u)(u^3-u^6)(u^6+u^5+u^3) & (1-u)(u-u^4) \\ 0 & 2(1+u+u^6) & 2(u^2+u^3-u^6) \\ 0 & u^6-u^5-u^3 & u+u^2+u^4-u^5 \end{pmatrix},$$

where $\alpha = \frac{1}{2(1-u)(u^3-u^6)}$. In the representation of $\Gamma(3,3,4;7)$ given in 3.1, conjugation by C acts as follows: $C^{-1}A_1C = I_1$, $C^{-1}PC = I_{321}$, $C^{-1}R_1^2PR_1^{-2}C = I_{1232123212}I_{321}I_{2123212321}$ (up to projective rescaling) and $C^*H_LC = H$. This calculation can be checked quickly using Maple. □

Lemma 3.2.4 *The word $I_{21231231213212}$ is contained in*

$$G = \langle I_1, I_{321}, I_{1232123212}I_{321}I_{2123212321} \rangle.$$

Proof: This follows from a straightforward word manipulation

$$I_{21231231213212} = I_1I_{123}I_{123}I_1I_{1232123212}I_{321}I_{2123212321}I_{123}.$$

□

Lemma 3.2.5 *$G=\langle I_1, I_{321}, I_{1232123212}I_{321}I_{2123212321}\rangle$ is an index 8 subgroup of $\Gamma(3,3,4;7)$.*

Proof: We show this by producing a left coset decomposition of the group $\Gamma(3,3,4;7)$. We will that prove every word in $\Gamma(3,3,4;7)$ lies in exactly one of the following left cosets,

$$G,\ I_2G,\ I_{12}G,\ I_{212}G,\ I_{3212}G,\ I_{13212}G,\ I_{213212}G,\ I_{1213212}G.$$

Consider the words of length 1, that is I_1, I_2 and I_3. Clearly I_1 lies in G and I_2 is contained in I_2G, so we only need to check I_3. Since $I_3 = I_2 \cdot I_{23}$ and $I_{23} \in G$, it follows that $I_3 \in I_2G$.

Length 2 words. We don't need to consider words that end with I_1 as we can reduce these to length one words by multiplying by I_1 on the right. This leaves I_{12}, I_{13}, I_{23} and I_{32}. The words I_{23} and I_{32} are contained in G and I_{12} is contained in $I_{12}G$. The only remaining word is $I_{13} = I_{12} \cdot I_{23}$, so $I_{13} \in I_{12}G$.

Length 3 words. Again we don't need to consider words that end in I_1, I_{32} or I_{23}, this only leaves I_{212}, I_{213}, I_{312}, I_{313}. Since $I_{213} = I_{212} \cdot I_{23}$, the words I_{212} and I_{213} both lie in $I_{212}G$. Similarly since $I_{313} = I_{12} \cdot I_{23} \cdot I_1$ and $I_{312} = I_{313} \cdot I_{32}$, both I_{312} and I_{313} lie in $I_{12}G$.

Length 4 words. From the arguments above it is clear that the only length four words not already contained in one of the previously described cosets are $I_1 I_{212}$ and $I_3 I_{212}$, since any other word can be immediately shortened to a word of length three or less by right multiplication by some $g \in G$. Since $I_{1212} \cdot I_1 = I_{212}$, the word I_{1212} lies in $I_{212}G$ and clearly $I_{3212} \in I_{3212}G$.

Length 5 words. We only need to consider the words $I_1 \cdot I_{3212}$ and $I_2 \cdot I_{3212}$. We use the relation $I_{23212} \cdot I_{23} \cdot I_1 \cdot I_{32} = I_{3212}$ to show $I_{23212} \in I_{3212}G$ and clearly $I_{13212} \in I_{13212}G$.

Length 6 words. We only need to consider the words I_{213212} and I_{313212}. We use the relation $I_{313212} \cdot I_1 = I_{13212}$ to show $I_{313212} \in I_{13212}G$ and clearly $I_{213212} \in I_{213212}G$.

Length 7 words. We only need to consider the words $I_{1213212}$ and $I_{3213212}$. We use the relation $I_{3213212} \cdot I_{21231231213212} = I_{1213212}$ (lemma 3.2.4) so both $I_{1213212}$, $I_{3213212} \in I_{1213212}G$.

Length 8 words. We only need to consider the words $I_{21213212}$ and $I_{31213212}$. Using the word from lemma 3.2.4 we can simplify these words to $I_{1213212}$ and I_{213212} respectively, so we do not need to introduce any additional cosets. Furthermore this means that all words of length eight or longer can be reduced to one of the eight coset representative words via right multiplication by some $g \in G$.

We must now show that a word lies in exactly one coset. This can be done by checking that the product $g_i^{-1} g_j$, of any two coset representatives g_i and g_j is not in

G. For example $I_{1213212}^{-1}I_{213212} = I_{2123121213212}$ and

$$I_2 = I_{123} \cdot I_{132} \cdot I_{123} \cdot I_{132} \cdot I_{2123121213212} \cdot I_{231} \cdot I_{321} \cdot I_{231} \cdot I_{321}.$$

A similar process of left and right multiplication by words in G can be carried out for all pairs of coset representatives, g_i and g_j, to reduce the product, $g_i^{-1}g_j$, to I_2. There are two possibilities, if $I_2 \in G$, then G is the whole group and all the cosets are the same. Alternatively, if $I_2 \notin G$ then all the cosets are disjoint. $\square$

Rather than show that $I_2 \notin G$, we show that $CI_2C^{-1} \notin \Gamma(7, \frac{3}{14})$ which implies the equivalent condition, that $CI_2C^{-1} \notin G' = \langle A_1, P, R_1^2PR_1^{-2} \rangle$.

Lemma 3.2.6 $CI_2C^{-1} \notin \Gamma(7, \frac{3}{14})$.

Proof: First observe that

$$CI_2C^{-1} = \begin{pmatrix} 0 & \dfrac{-u-u^3-u^4}{2} & -u-u^2 \\ \dfrac{u^5+1+u^4}{(u-1)^2} & \dfrac{u-u^2-1}{(u-1)^2} & \dfrac{u^5-u^4-1}{(u-1)^2} \\ \dfrac{u^5+1+u^3}{(u-1)^2} & \dfrac{-u-u^4+u^5-u^6}{2(u-1)^2} & \dfrac{u}{(u-1)^2} \end{pmatrix}. \tag{3.2}$$

We cannot simplify these fractions any further, in particular the denominator of the terms in the bottom two rows cannot be reduced to $(u-1)$. To see this let F_u be a polynomial in u, then $(u-1)$ is a factor of $F(u)$ iff $F_u = 0$ when we let $u = 1$, furthermore $F_u = F_u + 1 + u + u^2 + u^3 + u^4 + u^5 + u^6$. Using these two results we observe that $(u-1)$ is a factor of F_u iff $\Sigma_{i=0}^{6} a_i \equiv 0 \bmod 7$, where a_i are the coefficients of u^i un F_u. For a polynomial in u, F_u, we denote $\Sigma_{i=0}^{6} a_i$ by S_F.

Examining the polynomials in the numerators in the entries of CI_2C^{-1}, we see that they have S_F not equal to 0 in the bottom two rows, so we cannot remove a factor of $(u-1)$ from these entries. Note that we have used the fact that $\mathbb{Z}[u]$ is a unique factorisation domain.

Claim: In the presentation given above, the entries of any matrix in $\Gamma(7, \frac{3}{14})$ are rational polynomials in u divided by, at worst, a factor of $(u-1)$.

Since some the entries of CI_2C^{-1} cannot be written as a polynomial in u divided by $(u-1)$, it follows CI_2C^{-1} is not in $\Gamma(7, \frac{3}{14})$ and therefore I_2 is not in H. $\square$

Proof:[of claim] First we observe that P^n has this property for all n. In principle we could check this for all values of n but it suffices only to check that

$$P^2 = \frac{1}{2}\begin{pmatrix} u+u^2 & u^3-u^2 & u-u^2 \\ -2u^4/(u-1) & -2u^4/(u-1) & 2u^4/(u-1) \\ -(u+u^3)/(u-1) & -(u^2+u^4)/(u-1) & (-u+u^3+2u^2)/(u-1) \end{pmatrix}$$

and

$$P^3 = \frac{1}{2}\begin{pmatrix} 1-u+u^5+u^6 & 0 & -1+u+u^5-u^6 \\ 0 & -2u & 0 \\ 1+u+u^5+u^6 & 0 & -1-u-u^6+u^5 \end{pmatrix}$$

When checking these matrices, there are a number of somewhat suprising relations that help us to clear the denominator e.g $(u^4+u^2+1)^{-1} = -(u+u^2)$ and $(u^5+u^2+1)^{-1} = -(u^3+u^4)$.

Since P^3 has nothing in the numerator and P and P^2 both have at worst $1/2(u-1)$, we can write out P^n as $P^{3m}\cdot P$ or $P^{3m}\cdot P^2$ to see that the claim holds for all powers of P. Recall that since $PR_1P = P^3R_1^{-1} = R_1^{-1}P^3$ and R_1^n and P^3 both have nothing in the denominator it follows PR_1P will also have this property. We can extend this argument to all words in the group. □

Lemma 3.2.7 *$\langle A_1, P, R_1^2PR_1^{-2}\rangle$ is an index 8 subgroup of $\Gamma(7,\frac{3}{14})$.*

Proof: Let $G' = \langle A_1, P, R_1^2PR_1^{-2}\rangle$, we can partition $\Gamma(7,\frac{3}{14})$ into the following left cosets G', R_1G', R_1^2G', R_1^3G', R_1^4G', R_1^5G', R_1^6G', PR_1^3G'.

We want show that given any word in $\Gamma(7,\frac{3}{14})$ we can, by right multiplication by G', send it to one of the eight coset words. If a word ends with either A_1 or P, we can immediately shorten those words, since A_1 and P are in G'. So we may assume our word ends in a non-trivial power of R_1. Furthermore since A_1 commutes with R_1 we can assume the word ends in PR_1^n. These words can be further simplified by left multiplication by a word in G' as follows

$$\begin{aligned} PR_1 \cdot (P^{-2}) &= R_1^6, \\ PR_1^2 \cdot (P^{-2}) \cdot (R_1^2 P^{-2} R_1^{-2}) &= R_1^3, \\ PR_1^4 \cdot (R_1^2 P^2 R_1^{-2}) \cdot (P^{-1}) &= R_1, \\ PR_1^5 \cdot (R_1^2 P^{-1} R_1^{-2}) &= R_1^5, \\ PR_1^6 \cdot (R_1^2 P^{-2} R_1^{-2}) &= R_1^4. \end{aligned}$$

Whenever a word ends in PR_1^n, $n \neq 3$, we can change it into a word that ends in R_1^m by right multiplication by some word in G'. When a word ends in $R_1^n PR_1^3$ we can use the following relations to simplify the word,

$$R_1^n PR_1^3 \cdot (R_1^2 P^2 R_1^{-2} P^2) = R_1^{n+1} PR_1^3, \tag{3.3}$$

since R_1 has finite order, we can repeatedly use this relation to remove any power of R_1 on the left of PR_1^3. When a word ends in $P^n PR_1^3$ we can use the following relations to simplify the word,

$$P^n PR_1^3 \cdot (R_1^2 P^2 R_1^{-2}) \cdot (P^{-1}) = P^{n-1} R_1^2, \tag{3.4}$$

then using the relation $PR_1^2 \rightarrow R_1^3$ above, this becomes

$$P^{n-2} PR_1^2 \cdot (P^{-2}) \cdot (R_1^2 P^{-2} R_1^{-2}) = P^{n-2} \cdot R_1^3 = P^{n-3} \cdot PR_1^3. \tag{3.5}$$

Since P has finite order, we can repeatedly use this relation to remove any power of P on the left of PR_1^3. Finally we have

$$A_1 PR_1^3 \cdot (A_1) \cdot (R_1^2 P^2 R_1^{-2}) \cdot (A_1) \cdot (P^{-3}) \cdot (R_1^2 P R_1^{-2}) \cdot (A_1) = PR_1^3. \tag{3.6}$$

Combining these relations, any word in $\Gamma(7, \frac{3}{14})$ can be sent to one of the eight coset words by right multiplication by some $h' \in G'$. We also have to check that any word in $\Gamma(7, \frac{3}{14})$ lies in exactly one coset, as before checking the product $g_i^{-1} g_j$, of any two coset representatives g_i and g_j does not lie in G' is sufficient. By left and right multiplication by words in G' we can always send this product to R_1^n for $1 \leq n \leq 6$, using relation (3.3) above where necessary.

To see that non-trivial powers of R_1 are not contained in G' we argue as follows, the images of the generators of G' under conjugation by C are $C^{-1}A_1C = I_1$, $C^{-1}PC = I_{321}$, $C^{-1}R_1^2PR_1^{-2}C = I_{1232123212}I_{321}I_{2123212321}$. These three matrices all have entries in $\mathbb{Z}[e^{2i\pi/7}]$, so the image of any word in G', under conjugation by C, will also have entries in $\mathbb{Z}[e^{2i\pi/7}]$. However for any non-trivial power of R_1, the matrix $C^{-1}R_1^nC$, does not have all its entries in this ring. In particular the entries of bottom-right two-by-two sub-matrix are of the form $F/(1-e^{2i\pi/7})$, where is F some polynomial in $\mathbb{Z}[e^{2i\pi/7}]$ not divisible by $(1-e^{2i\pi/7})$. Therefore non-trivial powers of R_1 are not contained in G' an the 8 cosets are distinct. □

From lemmas 3.2.3, 3.2.5 and 3.2.7 it follows that $\Gamma(3,3,4;7)$ and $\Gamma(7,\frac{3}{14})$ are commensurable, moreover the groups have the same covolume.
We now show $\Gamma(3,3,5;5)$ and $\Gamma(10,0)$ are commensurable.

Lemma 3.2.8 *Consider the subgroups*

$$\langle I_1, I_{321}, I_{1232123212}I_{321}I_{2123212321}\rangle \subset \Gamma(3,3,4;7)$$

and

$$\langle A_1, P, R_1^2PR_1^{-2}\rangle \subset \Gamma(7,\tfrac{3}{14})$$

There is matrix C such that $C^{-1}A_1C = I_1$, $C^{-1}PC = I_{321}$, $C^{-1}R_1^2PR_1^{-2}C = I_{32131213}I_{321}I_{31213123}$ (up to projective rescaling), and $C^*H_LC = H$. So $\Gamma(3,3,5;5)$ and $C^{-1}\left(\Gamma(10,0)\right)C$ share a common subgroup namely $\langle I_1, I_{321}, I_{32131213}I_{321}I_{31213123}\rangle$.

Proof: Let $u = e^{2i\pi/5}$. In [22], Parker gives the following representation for $\Gamma(10,0)$ (recall $e^{2i\pi/10} = -(e^{2i\pi/5})^3$)

$$R_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -u^3 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad A_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$P = \frac{1}{2(1+u^2)}\begin{pmatrix} -u^3+u^2 & -u^3-u^2-2 & 2+u^3+u^2 \\ 2u^3 & -2 & -2u^3 \\ u^3+u^2 & u^3+u^2 & 2-u^3+u^2 \end{pmatrix}$$

preserving the Hermitian form

$$H_L = (-u^3+u^2)\begin{pmatrix} -1/(2+u^3+u^2) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/(-u^3-u^2) \end{pmatrix}$$

The matrix that, by conjugation, maps the representation of the first group to the second is

$$C = \begin{pmatrix} 1 & u^4+u^3 & u-u^4-u^3+u^2 \\ 0 & -u+u^4 & 2u+2u^2 \\ 0 & u^4 & -2u^4 \end{pmatrix}$$

In the representation for $(3,3,5;5)$ described in section 3.1, conjugation by C acts as follows, $C^{-1}A_1C = I_1$, $C^{-1}PC = I_{321}$, $C^{-1}R_1^2PR_1^{-2}C = I_{32131213}I_{321}I_{31213123}$, and $C^*H_LC = H$. Therefore the subgroups are the same. □

Lemma 3.2.9 *The word $I_{21231212}$ is contained in the subgroup*

$$G := \langle I_1, I_{321}, I_{32131213}I_{321}I_{31213123}\rangle \subset \Gamma(3,3,5;5). \tag{3.7}$$

Proof: This follows from a straightforward word manipulation,

$$I_{21231212} = I_{123}I_{123}I_{32131213}I_{321}I_{31213123}I_{321}I_{321}I_1.$$

□

Lemma 3.2.10 *$\langle I_1, I_{321}, I_{32131213}I_{321}I_{31213123}\rangle$ is an index 5 subgroup of $\Gamma(3,3,5;5)$.*

Proof: This follows along exactly the same lines as the proof of lemma 3.2.5. Let $G = \langle I_1, I_{321}, I_{32131213}I_{321}I_{31213123}\rangle$, the left cosets are

$$G,\ I_2G,\ I_{12}G,\ I_{212}G,\ I_{3212}G.$$

Proceeding as in the proof of lemma 3.2.5 once we reach length 5 words we can use the word $I_{21231212}$ from lemma 3.2.9 to terminate the procedure. □

Lemma 3.2.11 *The word R_1^2 is contained in the subgroup $\langle A_1, P, R_1PR_1^{-1}\rangle \subset \Gamma(10,0)$.*

Proof: This follows from a straightforward word manipulation,

$$R_1^2 = (R_1PR_1^{-1})(R_1PR_1^{-1})P^{-1}.$$

$\square$

Lemma 3.2.12 *$\langle A_1, P, R_1PR_1^{-1}\rangle$ is an index two subgroup of $\Gamma(10,0)$.*

Proof: Let $G' = \langle A_1, P, R_1PR_1^{-1}\rangle$, then we can partition $\Gamma(10,0)$ into the following cosets.

$$G',\ R_1G'.$$

Proceeding as in the proof of lemma 3.2.7, we show that any word can be sent to either the identity or R_1, by right multiplication by $h \in G'$. As before we can assume the word ends with some power of R_1 since A_1 and P are contained in G'. By lemma 3.2.11 R_1^2 is contained in G' so we may assume our word ends with R_1 (recall R_1 has order 10 in $\Gamma(10,0)$). Since A_1 commutes with R_1, we can always shorten a word that ends with A_1R_1 to word ending with R_1. That only leaves words ending with P^nR_1. We have the following relation,

$$P^nR_1 \cdot P^{-2} \cdot R_1^2 = P^{n-1} \cdot R_1$$

Since P has finite order, repeated use of this relation allows us to remove any power of P. Therefore any word in $\Gamma(10,0)$ can be reduced to either Id or R_1 by right multiplication by some $h \in G'$. $\square$

Lemmas 3.2.8, 3.2.10 and 3.2.12 together prove that $\Gamma(3,3,5;5)$ and $\Gamma(10,0)$ are commensurable and that the common subgroup is index 5 in $\Gamma(3,3,5;5)$ and index 2 in $\Gamma(10,0)$.

Lemma 3.2.13 ([29] or section 5.2 of [23]) *The covolume of a Livné group is given by*

$$\left(\frac{p-5}{2p^2}\right)\left(\frac{8\pi^2}{3}\right)$$

where p is the order of R_1 in the group.

Corollary 3.2.14 *The lattices* $\Gamma(7, \frac{3}{14})$ *and* $\Gamma(10,0)$ *have covolumes* $8\pi^2/147$ *and* $8\pi^2/120$ *respectively.*

Proposition 3.2.15 *The lattice* $\Gamma(3,3,4;7)$ *has covolume* $8\pi^2/147$ *and the lattice* $\Gamma(3,3,5;5)$ *has covolume* $8\pi^2/300$.

PROOF: The lattices $\Gamma(7, \frac{3}{14})$ and $\Gamma(3,3,4;7)$ share a common subgroup of index 8 in both groups. Therefore

$$\mathrm{vol}(\Gamma(3,3,4;7)) = \mathrm{vol}(\Gamma(7, \tfrac{3}{14})) = \frac{8\pi^2}{147}.$$

The lattices $\Gamma(10,0)$ and $\Gamma(3,3,5;5)$ share a common subgroup of index 2 and index 5 respectively. Therefore

$$\mathrm{vol}(\Gamma(3,3,5;5)) = \tfrac{2}{5}\mathrm{vol}(\Gamma(10,0)) = \frac{8\pi^2}{300}.$$

$\square$

Corollary 3.2.16 *Using the identifications described in chapter 2, the following triangle groups are lattices* $\Gamma(3,4,7;3)$, $\Gamma(3,3,7;7)$, $\Gamma(3,4,7;3)$ *and* $\Gamma(7,7,14;4)$ *since all these groups are identified with* $\Gamma(3,3,4;7)$ *as subgroups in* $\mathbf{PU}(2,1)$. *Also the groups* $\Gamma(3,5,5;5)$ *and* $\Gamma(5,5,10;3)$ *are lattices since they are both are identified with* $\Gamma(3,3,5;5)$ *as subgroups in* $\mathbf{PU}(2,1)$.

3.2.1 Presentations

Since we have presentations for the Livné groups and know the coset permutation table for the common subgroup, we can use the Reidemeister-Schreier method to obtain presentations for $\Gamma(3,3,4;7)$ and $\Gamma(3,3,5;5)$. Further details about the techniques used in this section can be found in most books on combinatorial group theory, for example [16]. We will only go through the details for $\Gamma(3,3,4;7)$ since the procedure is the same for both groups. Let $K = \Gamma(7, \frac{3}{14})$, using the left coset decomposition, we get a Shreier transversal for the subgroup G'

$$U = \left\{Id, R_1, R_1^2, R_1^3, R_1^4, R_1^5, R_1^6, PR_1^3\right\}.$$

Then the Reidemeister-Schreier method produces a presentation for G'. After simplifying by Tietze transformations, we get the following presentation:

$$\left\langle A_1, P, R_1^2 P R_1^{-2} \;\middle|\; \begin{array}{l} A_1^2,\ P^{42},\ (R_1^2 P R_1^{-2})^{42},\ (PA_1)^3,\ (R_1^2 P R_1^{-2} A_1)^3, \\ (R_1^2 P R_1^{-2})^2 A_1 P^2 A_1 P (R_1^2 P R_1^{-2})^{-2} A_1,\ P^3 (R_1^2 P R_1^{-2})^{-3} \end{array} \right\rangle.$$

Using the conjugation between the subgroups, we obtain the following relations for $G = \langle I_1, I_{321}, I_{1232123212} I_{321} I_{2123212321} \rangle$,

$$\begin{aligned} &I_1^2, \quad I_{321}^{42}, \quad (I_{1232123212} I_{321} I_{2123212321})^{42}, \quad I_{23}^3, \quad (I_{1232123212} I_{321} I_{2123212321} I_1)^3, \\ &(I_{1232123212} I_{321} I_{2123212321})^2 I_1 (I_{321})^2 I_1 I_{321} (I_{1232123212} I_{321} I_{2123212321})^{-2} I_1, \\ &I_{321}^3 (I_{1232123212} I_{321} I_{2123212321})^{-3} \end{aligned} \tag{3.8}$$

Since we know the coset permutations for G in $K = \Gamma(3,3,4;7)$ and that K is generated by I_1, I_2 and I_3, we can use a 'reverse' Reidemeister-Schreier method to obtain a presentation for $\Gamma(3,3,4;7)$. This formulation is taken from a correspondence with Professor Derek Holt, but it follows from a close reading of the standard Reidemeister-Schreier method.

'Reverse' Reidemeister-Schreier

Proposition 3.2.17 *Let $\langle Y|S \rangle$ be a presentation for $G \subset K = \langle X|R \rangle$. Assume we know Y, S and X, i.e. we have a presentation for G and we know the generators for K but not the relations. In addition assume we know a (left) transversal $U = \{u_i\}$ for G in K and the permutation relations for the u_i.*

For each $x \in X$ and $u_i \in U$, there is a relation of the form $x \cdot u_i = u_j \cdot h_{i,x}$, where u_j is another word in U and $h_{i,x}$ is a word in $\langle Y \rangle$. Then we can obtain a new presentation for K, with generating set $X \cup Y$ and relations

1. *For each $y \in Y$, a relation of the form $y = w$ where $w \in \langle X \rangle$,*
2. *All the $xu_i = u_j h_{i,x}$ relations described above and*
3. *The defining relations for G (i.e. the set S).*

Applying this process to the generators I_1, I_2, I_3 and Schreier transversal

$$U = \{Id, I_2, I_{12}, I_{212}, I_{3212}, I_{13212}, I_{213212}, I_{1213212}\},$$

we obtain the extra relations $I_2^2 = I_3^2 = I_{12}^4 = I_{13}^3 = Id$ as follows: let $x = I_2$ and $u_i = I_2$ as above, then $u_j = Id$ and $h_{i,x} = Id$, from which we obtain $I_2^2 = Id$. If $x = I_3$ and $u_i = I_{3212}$, then $u_j = I_{212}$ and $h_{i,x} = Id$, which gives the relation $I_3^2 = Id$. There are similar relations for the other two relations. We can use these new relations to simplify the group relations of (3.8) for example, $Id = (I_{1232123212}I_{321}I_{2123212321})^2 I_1 (I_{321})^2 I_1 I_{321} (I_{1232123212}I_{321}I_{2123212321})^{-2} I_1$, simplifies as follows

$$
\begin{aligned}
Id &= 1232123212321321212321232\underline{11}32132\underline{11}32\underline{11}232123212123123212321232\underline{11} \\
&= 1232123212321321212321\underline{23232}13\underline{232232}123212123123212321232 \\
&= 12321232123213212123 2\underline{13131}2321212312321232 1232 \\
&= 1232123212321321 21\underline{2323232}12123123212321232 \\
&= 12321232123213\underline{212121212}3123212321232 \\
&= 1232123212321\underline{323}123212321232 \\
&= 1232123212321232123212321232 \\
&= 1232^7
\end{aligned}
$$

By similar calculations the other relations listed in (3.8) simplify to give I_{1323}^7, I_1^2 or I_{123}^{42}. Clearly the group is generated by I_1, I_2 and I_3, thus we have the following presentation for $\Gamma(3,3,4;7)$.

A presentation for $\Gamma(3,3,4;7)$

$$
\langle I_1, I_2, I_3 \mid I_i^2,\ I_{23}^3,\ I_{31}^3,\ I_{12}^4,\ I_{1323}^7,\ I_{123}^{42} \rangle. \tag{3.9}
$$

Exactly the same process can be carried out to obtain a group presentation for $\Gamma(3,3,5;5)$.

A presentation for $\Gamma(3,3,5;5)$

$$
\langle I_1, I_2, I_3 \mid I_i^2,\ I_{23}^3,\ I_{31}^3,\ I_{12}^5,\ I_{1323}^5,\ I_{123}^{15} \rangle. \tag{3.10}
$$

3.3 Fundamental Domains

Having shown that $\Gamma(3,3,4;7)$ and $\Gamma(3,3,5;5)$ are lattices, is this section we describe conjectural fundamental domains, Δ, for the groups. The fundamental domains are essentially the same, the same construction could be carried out for any $\Gamma(p, q, r; n)$ group with I_{123} finite order regular elliptic. Obviously if the group is non-discrete this will not lead to a fundamental domain. The fundamental domain consists of a simply connected bounded 3-dimensional region of $\mathbf{H}^2_{\mathbb{C}}$, over which we then take the four dimensional cone to the fixed point of I_{123}.

These fundamental domains are conjectural since we only define them combinatorially and we assume that they satisfy all the conditions of Poincaré's polyhedron theorem. These domains have the correct covolume and produce the same group presentations, so we believe they could be modified to be true fundamental domains via the same method used in the next chapter for $\Gamma(4, 4, 4; 5)$.

The core faces

We begin the construction of our fundamental domain by defining **core faces**, these are three dimensional polyhedra. The first core face depends only on $p = \mathrm{ord}(I_{23})$. Consider the p-gon with the following vertices 12, $12.23 = 13$, $12.2323 = 1323, \ldots,$ $12.(23^{p-1})$. Note that (23) has order p, so $12.(23)^p = 12$. All these points lie in the complex geodesic fixed by I_1, we denote this line $\mathcal{C}_1$. Then we take the suspension of this p-gon to the points 23 and 1231. Since $I_1(23) = 1231$ and the p-gon lies entirely in $\mathcal{C}_1$, it is clear, at least combinatorially, that this polyhedron is preserved by I_1. The second core face depends on $q = \mathrm{ord}(I_{13})$. Consider the q-gon with vertices $32.(13)^i$, again this polygon lies entirely in a complex geodesic, $\mathcal{C}_{323}$. We then take the suspension of the q-gon to the points 13 and $I_{323}(13) = 323123$. This face is preserved under I_{323}. The third core face is constructed the same way, a suspension of the r-gon $31.(13)^i$ to the points 12 and $I_3(12) = 3123$. This face is preserved under I_3. These are the polyhedra denoted **A**, **B** and **C** in figures 1 and 2.

The fourth core face is constructed by taking suspension of the ord(3212) sided polygon with vertices 23.3212^i to the points 3212 and 1232. This face is preserved under I_2. For $\Gamma(3, 3, 4; 7)$ and $\Gamma(3, 3, 5; 5)$ only one core piece is needed at this point in the construction, (this reflects the fact that in a $\Gamma(3, 3, r; n)$ group we can pass between all words of the form I_{ijkj} by conjugation and taking inverses), for other groups we would need more faces. These are polyhedra **D** in figures 1 and 2.

The combinatorics of the final pieces depends on the group, for $\Gamma(3,3,4;7)$ it is the tetrahedron with vertices 1232123212, 123212, 3212 and 31213212. This piece is preserved under $I_{2123212}$. For $\Gamma(3,3,5;5)$ it is a six faced polyhedron with vertices 123212, 2123212123, 12, 3212 and 312121. This piece is preserved under I_{212}. These are polyhedra **E** in figures 1 and 2.

Whenever two of these faces share a common triangle in their boundary, we say that they are glued along that triangle. In this way we can glue the five core faces to form a 3-dimension subset of $\mathbf{H}^2_{\mathbb{C}}$ that is homeomorphic to the 3-dimensional ball with a triangulated boundary. Then we take the affine cone over this region to the fixed point of I_{123}, denoted $*$. This gives us a 4-dimensional region, whose 3 dimensional boundary consists of the core pieces and the cones over their triangulated boundary.

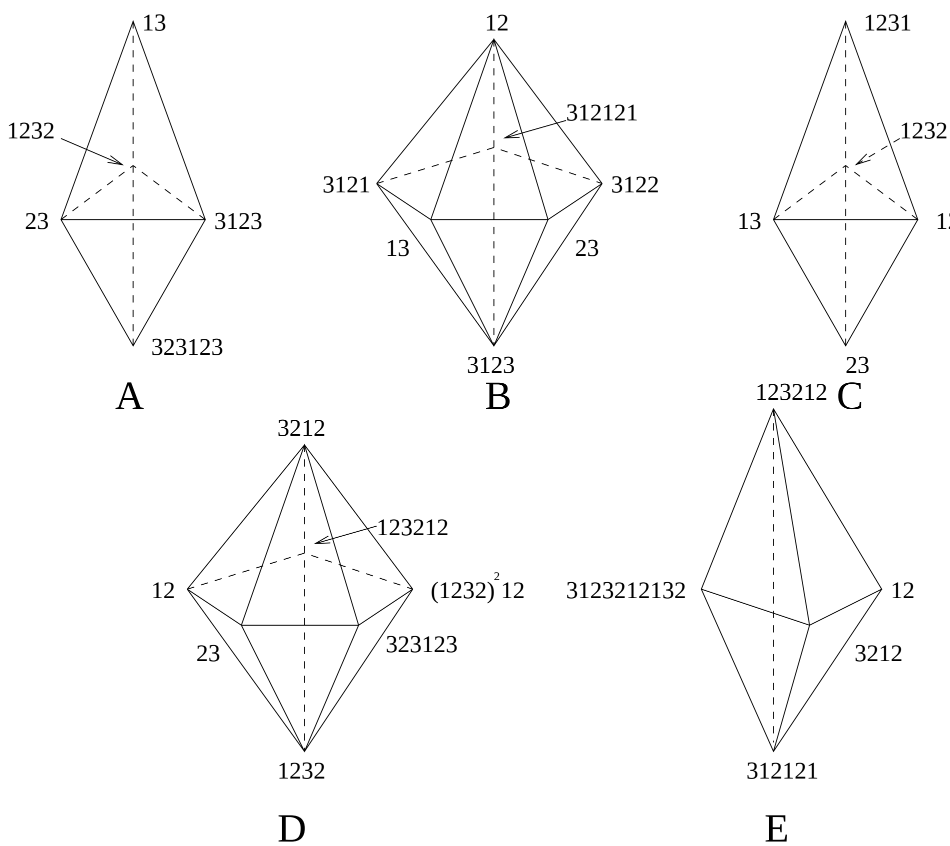

Figure 3.1: Core faces for $\Gamma(3,3,5;5)$

We denote this 4-dimensional polytope by Δ. This is a combinatorial fundamental domain for the group.

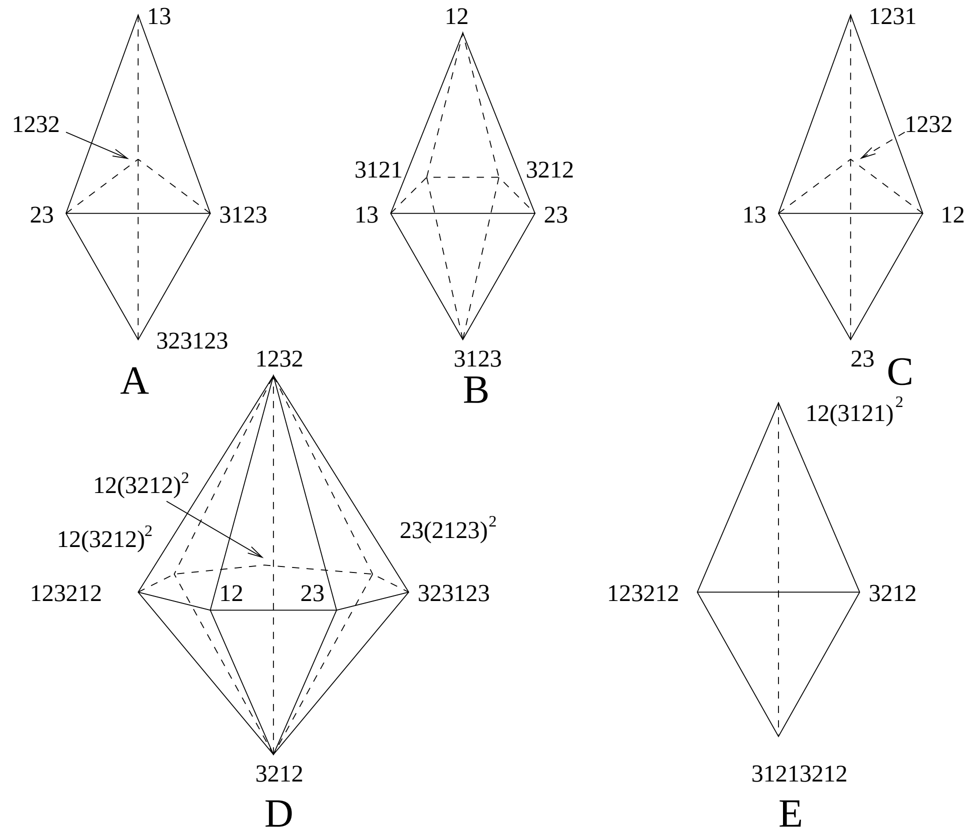

Figure 3.2: Core faces for $\Gamma(3, 3, 4; 7)$

We will simplify the construction by combining any cone faces that have the same side pairing transformation when we use Poincaré's polyhedron theorem.

Combined cone faces

Combined cone faces for $\Gamma(3,3,5;5)$.

$$\begin{aligned}
\mathbf{F}_1 =& \mathrm{Cone}(12, 1232, 123212) \cup \mathrm{Cone}(12, 1232, 1231) \cup \mathrm{Cone}(12, 1231, 13) \\
& \cup \mathrm{Cone}(12, 13, 3121) \cup \mathrm{Cone}(13, 3121, 3123) \cup \mathrm{Cone}(13, 3123, 1232) \\
& \cup \mathrm{Cone}(13, 1232, 1231), \\
\mathbf{F}_2 =& \mathrm{Cone}(3123, 3212, 312121) \cup \mathrm{Cone}(3123, 3212, 23) \cup \mathrm{Cone}(3123, 23, 323123) \\
& \cup \mathrm{Cone}(3123, 323123, 1232) \cup \mathrm{Cone}(323123, 1232, 1232123212) \\
& \cup \mathrm{Cone}(323123, 1232123212, 3212) \cup \mathrm{Cone}(323123, 3212, 23), \\
\mathbf{G}_1 =& \mathrm{Cone}(12, 3121, 312121) \cup \mathrm{Cone}(3121, 312121, 3123), \\
\mathbf{G}_2 =& \mathrm{Cone}(1232123212, 3212, 123212) \cup \mathrm{Cone}(3212, 123212, 2123212123), \\
\mathbf{H}_1 =& \mathrm{Cone}(1232123212, 123212, 1232), \\
\mathbf{H}_2 =& \mathrm{Cone}(2123212123, 312121, 3212), \\
\mathbf{I}_1 =& \mathrm{Cone}(12, 123212, 312121), \\
\mathbf{I}_2 =& \mathrm{Cone}(2123212123, 123212, 123212).
\end{aligned}$$

Note $I_{321}(\mathbf{F}_1) = \mathbf{F}_2$, $(I_{321})^2(\mathbf{G}_1) = \mathbf{G}_2$, $I_{321}(\mathbf{H}_1) = \mathbf{H}_2$ and $(I_{321})^3(\mathbf{I}_1) = \mathbf{I}_2$.

Combined cone faces for $\Gamma(3,3,4;7)$.

$$
\begin{aligned}
\mathbf{F}_1 = &\mathrm{Cone}(13, 1232, 3123) \cup \mathrm{Cone}(13, 3123, 3121) \cup \mathrm{Cone}(13, 3121, 12)\\
&\cup \mathrm{Cone}(13, 12, 1231) \cup \mathrm{Cone}(13, 1231, 1232) \cup \mathrm{Cone}(1232, 1231, 12)\\
&\cup \mathrm{Cone}(12, 1232, 123212),\\
\mathbf{F}_2 = &\mathrm{Cone}(323123, 3212, 23(2123)^2) \cup \mathrm{Cone}(323123, 23(2123)^2, 1232)\\
&\cup \mathrm{Cone}(323123, 1232, 3123) \cup \mathrm{Cone}(23123, 3123, 23) \cup \mathrm{Cone}(323123, 23, 3212)\\
&\cup \mathrm{Cone}(3212, 23, 3123) \cup \mathrm{Cone}(3123, 3212, 3121),\\
\mathbf{G}_1 = &\mathrm{Cone}(3121, 12, 3212) \cup \mathrm{Cone}(12, 3212, 123212) \cup \mathrm{Cone}(123212, 3212, 31213212),\\
\mathbf{G}_2 = &\mathrm{Cone}(3212, 23(2123)^2, 12(3212)^3) \cup \mathrm{Cone}(23(2123)^2, 12(3212)^3, 1232)\\
&\cup \mathrm{Cone}(1232, 12(3212)^3, 12(3212)^2),\\
\mathbf{H}_1 = &\mathrm{Cone}(12(3212)^2, 1232, 212321) \cup \mathrm{Cone}(12(3212)^2, 212321, 31213212),\\
\mathbf{H}_2 = &\mathrm{Cone}(12(3212)^2, 12(3212)^3, 3212) \cup \mathrm{Cone}(12(3212)^2, 3212, 31213212).
\end{aligned}
$$

Note, $I_{321}(\mathbf{F}_1) = \mathbf{F}_2$, $(I_{321})^2\,(\mathbf{G}_1) = \mathbf{G}_2$, and $(I_{321})^3\,(\mathbf{H}_1) = \mathbf{H}_2$.

These faces for $\Gamma(3,3,5;5)$ and $\Gamma(3,3,4;7)$ are shown in figures 3 and 4 respectively.

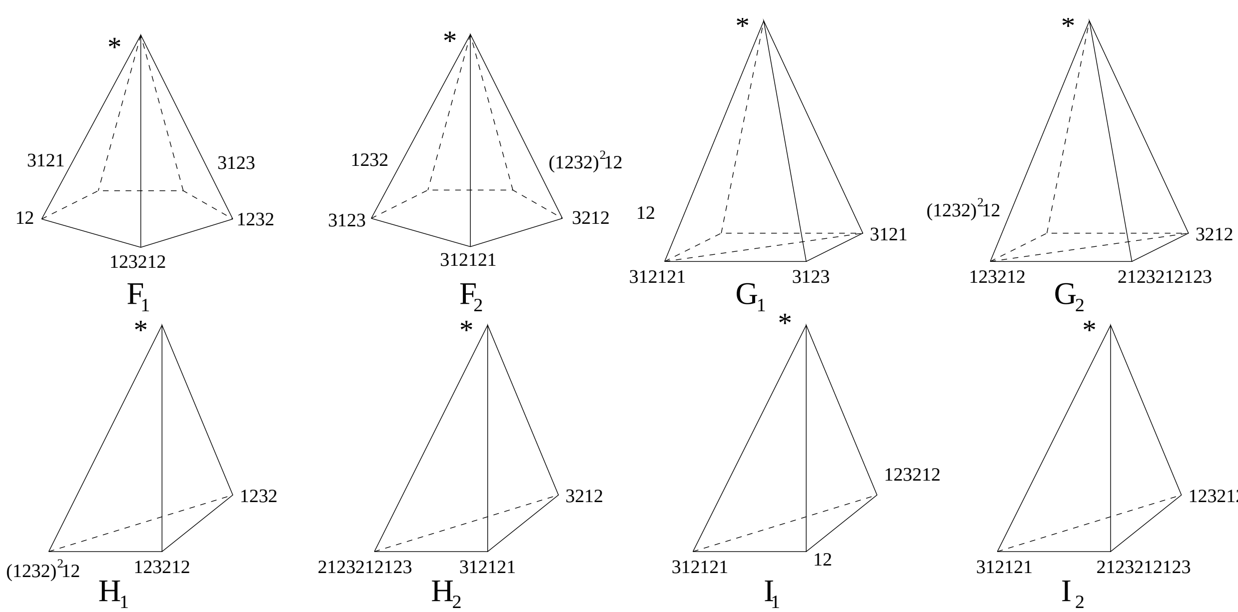

Figure 3.3: Cone faces for $\Gamma(3,3,5;5)$

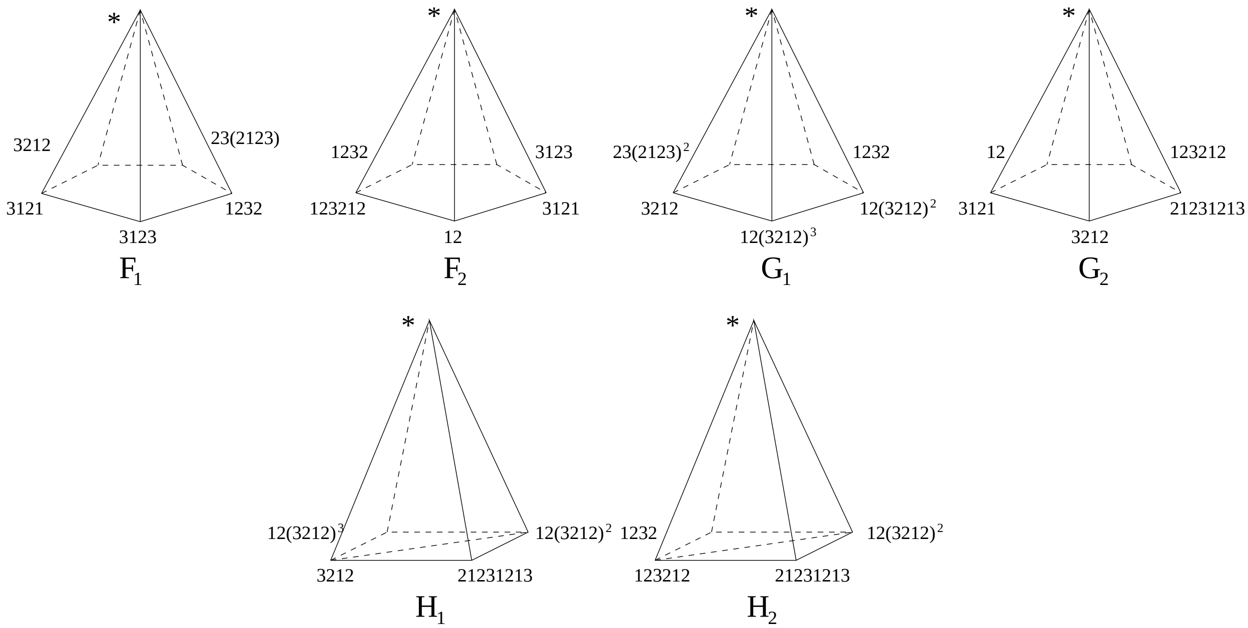

Figure 3.4: Cone faces for $\Gamma(3,3,4;7)$

3.3.1 Poincaré's polyhedron theorem and group presentations

In this section we use Poincaré's polyhedron theorem to obtain a presentation for the group, we assume all the necessary conditions are met. Rather than list the entire orbit for each cycle, we will list a representative 2-face, the length of the cycle and the word given by the cycle relation. We state Poincaré's polyhedron theorem fully in chapter 4, section 4.4.

3.3.2 Side pairing relations

Side pairing relations for $\Gamma(3,3,5;5)$

$$\begin{array}{ccc} I_{232}:\mathbf{A}\to\mathbf{A}, & I_3:\mathbf{B}\to\mathbf{B}, & I_1:\mathbf{C}\to\mathbf{C}, \\ I_2:\mathbf{D}\to\mathbf{D}, & I_{212}:\mathbf{E}\to\mathbf{E}, & I_{321}\mathbf{F}_1\to\mathbf{F}_2, \\ I_{321}^2\mathbf{G}_1\to\mathbf{G}_2, & I_{321}\mathbf{H}_1\to\mathbf{H}_2, & I_{321}^3\mathbf{I}_1\to\mathbf{I}_2. \end{array}$$

Side pairing relations for $\Gamma(3,3,4;7)$

$$I_{232} : \mathbf{A} \to \mathbf{A}, \quad I_3 : \mathbf{B} \to \mathbf{B}, \quad I_1 : \mathbf{C} \to \mathbf{C},$$
$$I_2 : \mathbf{D} \to \mathbf{D}, \quad I_{2123212} : \mathbf{E} \to \mathbf{E}, \quad I_{321}\mathbf{F}_1 \to \mathbf{F}_2,$$
$$I_{321}^2\mathbf{G}_1 \to \mathbf{G}_2, \quad I_{321}^3\mathbf{H}_1 \to \mathbf{H}_2.$$

3.3.3 Cycle relations and group presentations

Cycle relations for $\Gamma(3,3,5;5)$

We already have the reflection relations $11 = 22 = 33 = Id$, this allows to remove double letters in the cycle words to get the cycle relations shown in the table below

Face	Cycle length	Cycle relation
$(13, 1232, 23)$	4	Id
$(13, 23, 3123)$	4	$232323 = Id$
$(13, 1232, 3123)$	6	$23232131232321 = Id$
$(12, 23, 3212)$	4	Id
$(12, 3212, 312121)$	4	Id
$(12, 3121, 312121)$	6	$31231(2321)^3 321 = Id$
$(123212, 12, 312121)$	4	$212(123)^3 212(321)^3 = Id$
$(3121, 3123, *)$	4	Id
$(3121, 12, *)$	4	Id
$(3121, 312121, *)$	4	Id
$(123212, 1232, *)$	2	Id
$(12, 312121, *)$	3	Id
$(312121, 123212, *)$	1	$(321)^{15} = Id$

There are 5 non-trivial cycle relations, after some manipulation of the words it can be seen that these correspond to the relations I_{23}^3, I_{13}^3, I_{1232}^5, I_{12}^5 and I_{123}^{15}. Then by taking inverses or conjugates of these words we obtain all the relations shown in presentation (3.11) below.

Cycle relations for $\Gamma(3,3,4;7)$

We already have the reflection relations $11 = 22 = 33 = Id$, this allows to remove double letters in the cycle words to get the cycle relations shown in the table below

Face	Cycle length	Cycle relation
$(13, 23, 1232)$	4	Id
$(13, 3123, 23)$	4	$232323 = Id$
$(13, 1232, 3123)$	6	$23232131232321 = Id$
$(1231, 1232, 12)$	4	Id
$(12, 3212, 3121)$	6	$2121231232321321 = Id$
$(1232, 12(3212)^2, 123212)$	5	$123212(123)^2 2(321)^3 = Id$
$(12(3212)^2, 3212, 21231213)$	4	$(2123212)(321)^3(2123212)(123)^3 = Id$
$(3121, 3212, *)$	3	Id
$(3212, 23(2123)^2, *)$	3	Id
$(23(2123)^2, 1232, *)$	3	Id
$(1232, 12(3212)^2, *)$	4	Id
$(12(3212)^2, 21231213, *)$	1	$(123)^{42} = Id$

This time there are 6 non-trivial cycle relations, after some manipulation of the words it can be seen that these correspond to the relations I_{23}^3, I_{13}^3, I_{1232}^7, I_{12}^4 and I_{123}^{42} (one of the relations is redundant, it can be derived from the other five). Then by taking inverses or conjugates of these words we obtain all the relations shown in presentation (3.12) below.

Group presentations

Using Poincaré's polyhedron theorem we produce the following presentations for the groups.

A presentation for $\Gamma(3,3,5;5)$

$$\langle I_1, I_2, I_3 \mid I_i^2,\ I_{23}^3,\ I_{31}^3,\ I_{12}^5,\ I_{1323}^5,\ I_{123}^{15} \rangle \tag{3.11}$$

A presentation for $\Gamma(3,3,4;7)$

$$\langle I_1, I_2, I_3 \mid I_i^2,\ I_{23}^3,\ I_{31}^3,\ I_{12}^4,\ I_{1323}^7,\ I_{123}^{42} \rangle \tag{3.12}$$

These presentations agree with those obtained from the Reidemeister-Schreier method in proposition 3.2.1

3.3.4 Euler orbifold characteristics

Euler orbifold characteristic for $\Gamma(3,3,5;5)$

Contribution to Euler orbifold characteristic from 0-faces.

Orbit representative	Stabiliser	Order	Euler
23	$\langle I_2, I_3\rangle$	6	1/6
31	$\langle I_3, I_1\rangle$	6	1/6
12	$\langle I_1, I_2\rangle$	10	1/10
1232	$\langle I_1, I_{232}\rangle$	10	1/10
312121	$\langle I_3, I_{12121}, (I_{123})^3\rangle$	100	1/100
$*$	$\langle I_{123}\rangle$	15	1/15
			61/100

Contribution to Euler orbifold characteristic from 1-faces.

Orbit representative	Stabiliser	Order	Euler
$(13, 23)$	$\langle I_3 \rangle$	2	$-1/2$
$(13, 3123)$	$\langle I_{313} \rangle$	2	$-1/2$
$(13, 1232)$	$\langle I_1 \rangle$	2	$-1/2$
$(23, 1232)$	$\langle I_{232} \rangle$	2	$-1/2$
$(23, 3123)$	$\langle I_{232} \rangle$	2	$-1/2$
$(1232, 3123)$	$\langle I_{232} \rangle$	2	$-1/2$
$(12, 3212)$	$\langle I_{212} \rangle$	2	$-1/2$
$(12, 312121)$	$\langle I_{12121} \rangle$	2	$-1/2$
$(1232, 123212)$	$\langle I_{12321} \rangle$	2	$-1/2$
$(123212, 312121)$	$\langle I_{212}, (I_{123})^3 \rangle$	10	$-1/10$
$(12, *)$	Id	1	-1
$(1232, *)$	Id	1	-1
$(312121, *)$	$\langle (I_{123})^3 \rangle$	5	$-1/5$
			$-34/5$

Contribution to Euler orbifold characteristic from 2-faces.

Orbit representative	Stabiliser	Order	Euler
$(13, 1232, 3123)$	Id	1	1
$(13, 1232, 23)$	Id	1	1
$(13, 23, 3123)$	Id	1	1
$(12, 23, 3123)$	Id	1	1
$(12, 3212, 312121)$	Id	1	1
$(12, 3121, 312121)$	Id	1	1
$(123212, 12, 312121)$	$\langle I_{212} \rangle$	2	$1/2$
$(3121, 3123, *)$	Id	1	1
$(3121, 12, *)$	Id	1	1
$(123212, 1232, *)$	Id	1	1
$(12, 312121, *)$	Id	1	1
$(3123, 312121, *)$	Id	1	1
$(312121, 123212, *)$	$\langle (I_{123})^3 \rangle$	5	$1/5$
			$117/10$

Contribution to Euler orbifold characteristic from 3-faces.

Orbit representative	Stabiliser	Order	Euler
A	$\langle I_{232} \rangle$	2	$-1/2$
B	$\langle I_3 \rangle$	2	$-1/2$
C	$\langle I_1 \rangle$	2	$-1/2$
D	$\langle I_2 \rangle$	2	$-1/2$
E	$\langle I_{212} \rangle$	2	$-1/2$
F_1	Id	1	-1
G_1	Id	1	-1
H_1	Id	1	-1
I_1	Id	1	-1
			$-13/2$

Contribution to Euler orbifold characteristic from 4-faces.

Orbit representative	Stabiliser	Order	Euler
Δ	Id	1	1
			1

The Euler orbifold characteristic of $(3,3,5;5)$ is

$$\frac{61}{100} - \frac{34}{5} + \frac{117}{10} - \frac{13}{2} + 1 = \frac{1}{100}.$$

Euler orbifold characteristic for $\Gamma(3,3,4;7)$

Contribution to Euler orbifold characteristic from 0-faces.

Orbit representative	Stabiliser	Order	Euler
23	$\langle I_2, I_3 \rangle$	6	1/6
31	$\langle I_3, I_1 \rangle$	6	1/6
12	$\langle I_1, I_2 \rangle$	8	1/8
1232	$\langle I_1, I_{232} \rangle$	14	1/14
1232123212	$\langle I_2, I_{123212321}, (I_{123})^3 \rangle$	392	1/392
$*$	$\langle I_{123} \rangle$	42	1/42
			109/196

Contribution to Euler orbifold characteristic from 1-faces.

Orbit representative	Stabiliser	Order	Euler
$(13, 3123)$	$\langle I_3 \rangle$	2	$-1/2$
$(13, 23)$	$\langle I_3 \rangle$	2	$-1/2$
$(13, 1232)$	$\langle I_1 \rangle$	2	$-1/2$
$(23, 3123)$	$\langle I_{232} \rangle$	2	$-1/2$
$(23, 1232)$	$\langle I_{232} \rangle$	2	$-1/2$
$(1232, 3123)$	$\langle I_{232} \rangle$	2	$-1/2$
$(12, 3212)$	$\langle I_{212} \rangle$	2	$-1/2$
$(3212, 3121)$	$\langle I_3 \rangle$	2	$-1/2$
$(1232, 1232123212)$	$\langle I_{123212321} \rangle$	2	$-1/2$
$(1232123212, 2123121)$	$\langle I_{2123212}, (I_{123})^3 \rangle$	28	$-1/28$
$(*, 12)$	$\langle Id \rangle$	1	-1
$(*, 1232)$	$\langle Id \rangle$	1	-1
$(*, 1232123212)$	$\langle (I_{123})^3 \rangle$	14	$-1/14$
			$-185/28$

Contribution to Euler orbifold characteristic from 2-faces.

Orbit representative	Stabiliser	Order	Euler
$(13, 3123, 23)$	Id	1	1
$(13, 23, 1232)$	Id	1	1
$(13, 1232, 3123)$	Id	1	1
$(1231, 1232, 12)$	Id	1	1
$(12, 3212, 3121)$	Id	1	1
$(1232, 1232123212, 123212)$	Id	1	1
$(132123212, 3212, 21231213)$	$\langle I_{2123212} \rangle$	2	$1/2$
$(3121, 3212, *)$	Id	1	1
$(3212, 2321232123, *)$	Id	1	1
$(1232, 2321232123, *)$	Id	1	1
$(1232, 1232123212, *)$	Id	1	1
$(1232123212, 21231213, *)$	$\langle (I_{123})^3 \rangle$	14	$1/14$
			$74/7$

Contribution to Euler orbifold characteristic from 3-faces.

Orbit representative	Stabiliser	Order	Euler
A	$\langle I_{232} \rangle$	2	$-1/2$
B	$\langle I_3 \rangle$	2	$-1/2$
C	$\langle I_1 \rangle$	2	$-1/2$
D	$\langle I_2 \rangle$	2	$-1/2$
E	$\langle I_{2123212} \rangle$	2	$-1/2$
F_1	Id	1	-1
G_1	Id	1	-1
H_1	Id	1	-1
			$-11/2$

Contribution to Euler orbifold characteristic from 4-faces.

Orbit representative	Stabiliser	Order	Euler
Δ	Id	1	1
			1

The Euler orbifold characteristic of $\Gamma(3,3,4;7)$ is

$$\frac{109}{196} - \frac{185}{28} + \frac{74}{7} - \frac{11}{2} + 1 = \frac{1}{49}.$$

The covolume of a lattice is equal to its Euler orbifold characteristic multiplied by $8\pi^2/3$. Consequently the covolumes of $\Gamma(3,3,4;7)$ and $\Gamma(3,3,5;5)$ of the combinatorial fundamental domains are $8\pi^2/147$ and $8\pi^2/300$ respectively. This agrees with covolumes calculated from the commensurability results in the previous section. In the next chapter we construct a true fundamental domain of Deraux's lattice $\Gamma(4,4,4;5)$, in principle the same techniques could be applied to the combinatorial domains in this section, however since $\Gamma(3,3,4;7)$ and $\Gamma(3,3,5;5)$ lack the triple symmetry of Deraux's lattice the calculations are far more involved.

Remark: There seems to be connection between $\Gamma(3,3,4;7)$ and the fake projective plane $\mathcal{C}_{31}$ described in [25], both groups are arithmetic over the same algebraic field and have the same covolume. There is a similar relationship between $\Gamma(4,4,4;5)$ and $\mathcal{C}_2$ (see remark 4.7 for more details). This connection merits further investigation.

Chapter 4

$\Gamma(4, 4, 4; 5)$

4.1 Deraux's Lattice

In this chapter we analyse the deformed triangle group $\Gamma(4, 4, 4; 5)$. As mentioned in chapter 2 this is one of four known deformed triangle groups that are lattices.

The group was shown to be a lattice by Deraux in [6], his method involved constructing a Dirichlet domain for the subset of words of length 1,2,3 and 4. Let G be a group of isometries and $g \in G$, we define the half space centred a p with respect g as

$$\mathcal{H}_p(g) := \{z \in \mathbf{H}^2_{\mathbb{C}} | d(z, p) < d(z, g(p))\}. \tag{4.1}$$

The Dirichlet domain centred at p for $\Gamma(4, 4, 4; 5)$ is the intersection of all these half spaces

$$\Delta_p := \bigcap_{g \in G} \mathcal{H}_p(g). \tag{4.2}$$

Using this notation, the domain constructed by Deraux is

$$\Delta_{p,W} := \bigcap_{g \in W} \mathcal{H}_p(g). \tag{4.3}$$

where W is the subset of $\Gamma(4, 4, 4; 5)$ consisting of words of length 1,2,3 and 4. Deraux showed that $\Delta_{p,W}$ is bounded. Since $\Delta_{p,W}$ must contain a true Dirichlet domain for all of $\Gamma(4, 4, 4; 5)$ it follows that $\Gamma(4, 4, 4; 5)$ was cocompact and combining this with the fact that the group is arithmetic, $\Gamma(4, 4, 4; 5)$ is a lattice. Then, using this domain, Deraux applied Poincaré's polyhedron theorem to produce the following presentation for the group

$$\left\langle I_1, I_2, I_3 \mid I_i^2, (I_iI_j)^4, (I_iI_jI_k)^{10}, (I_iI_jI_kI_j)^5 \right\rangle \tag{4.4}$$

This presentation is somewhat conjectural since it assumes that there are no extra faces and cycle relations arising from longer words.

In this chapter we construct a fundamental domain for the whole group, produce a presentation that agrees with Deraux and calculate the covolume of the lattice. We try to argue synthetically wherever possible. The main result from this chapter is as follows.

Theorem 4.1.1 (Main Theorem) *The group* $\Gamma(4, 4, 4; 5)$ *is a cocompact lattice, with presentation*

$$\left\langle I_1, I_2, I_3 \mid I_i^2, (I_iI_j)^4, (I_iI_jI_k)^{10}, (I_iI_jI_kI_j)^5 \right\rangle \tag{4.5}$$

for each choice of $i, j, k \in \{1, 2, 3\}$

There is an 'extra' relation implicit in this presentation

Proposition 4.1.2 $(I_2I_3I_1I_3I_2I_3I_1I_3I_2I_3)^6 = Id$

Proof: We have

$$\begin{aligned}
(231)3231323 &= (231)^2(123212321)132 \\
&= (231)^2 23(2123212)32(132) \\
&= (231)^3 2(1213121)2(132)^2 \\
&= (231)^3 2312(1312131)2132(132)^2 \\
&= (231)^4 231(3132313)132(132)^3 \\
&= (231)^5(3132313)(132)^4 \\
&= (231)^5 2313231323(132)^5.
\end{aligned}$$

Hence (2313231323) commutes with $(231)^5$. Moreover, using the expressions above we have:

$$\begin{aligned}
(2313231323)^3 &= \big((231)^5(3132313)(132)^4\big)\big((231)^4 2(1312131)2(132)^3\big)\big(2313231323\big) \\
&= (231)^5 3(132)31(21312131)2(132)^2 3231323 \\
&= (231)^5 3(132)13(1312)^3(13212321323 \\
&= (231)^5 3(132)^2 131213113212321323 \\
&= (231)^5 3(132)^3 121231132121(132)^2 3 \\
&= (231)^5 3(132)^5 3.
\end{aligned}$$

Therefore

$$\begin{aligned}
(2313231323)^6 &= (231)^5(2313231323)^3(132)^5(2313231323)^3 \\
&= (231)^5(231)^5 3(132)^5 3(132)^5(231)^5 3(132)^5 3 \\
&= (231)^{10} 3(132)^{10} 3 = Id.
\end{aligned}$$

□

4.1.1 A representation for Deraux's lattice

Let $\omega = \frac{-1}{2} + \frac{i\sqrt{3}}{2}$ and $\phi = \frac{1}{2} + \frac{\sqrt{5}}{2}$, recall $\omega^2 = \overline{\omega}$, $\omega + \overline{\omega} = -1$ and $\phi^2 = \phi + 1$, then we have a representation,

$$I_1 = \begin{pmatrix} 0 & \overline{\omega}\phi & \overline{w}\sqrt{\phi} \\ \omega\phi & \phi & \phi\sqrt{\phi} \\ -\omega\sqrt{\phi} & -\phi\sqrt{\phi} & -\phi^2 \end{pmatrix},$$

$$I_2 = \begin{pmatrix} \phi & -\phi^2 & \phi\sqrt{\phi}\,(\omega + \overline{\omega}\phi) \\ -\phi^2 & \phi & -\phi\sqrt{\phi}\,(\omega + \overline{\omega}\phi) \\ -\phi\sqrt{\phi}\,(\overline{\omega} + \omega\phi) & \phi\sqrt{\phi}\,(\overline{\omega} + \omega\phi) & -\phi^3 \end{pmatrix},$$

$$I_3 = \begin{pmatrix} \phi^3 & \phi\overline{\omega} - 1 & \phi\sqrt{\phi}\,(\phi\overline{\omega} - 1) \\ \phi\omega - 1 & 0 & \phi\sqrt{\phi} \\ -\phi\sqrt{\phi}\,(\phi\omega - 1) & -\phi\sqrt{\phi} & -\left(\phi^3 + 1\right) \end{pmatrix}.$$

These matrices preserve the Hermitian form

$$H = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

From which we derive,

$$I_{131} = \begin{pmatrix} 0 & \omega\phi & -\sqrt{\phi} \\ \overline{\omega}\phi & \phi & -\overline{\omega}\phi\sqrt{\phi} \\ \sqrt{\phi} & \omega\phi\sqrt{\phi} & -\phi^2 \end{pmatrix},$$

$$I_{121} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$I_{212} = \begin{pmatrix} \phi & \omega\phi & -\phi\sqrt{\phi} \\ \overline{\omega}\phi & 0 & -\overline{\omega}\sqrt{\phi} \\ \phi\sqrt{\phi} & \omega\sqrt{\phi} & -\phi^2 \end{pmatrix},$$

$$I_{12321} = \begin{pmatrix} \phi & \overline{\omega}\phi & \overline{\omega}\phi\sqrt{\phi} \\ \omega\phi & 0 & \sqrt{\phi} \\ -\omega\phi\sqrt{\phi} & -\sqrt{\phi} & -\phi^2 \end{pmatrix},$$

$$I_{1232} = \begin{pmatrix} 0 & \overline{\omega} & 0 \\ -\omega\phi & 0 & -\sqrt{\phi} \\ \omega\sqrt{\phi} & 0 & \phi \end{pmatrix}.$$

It can easily be checked that these matrices are indeed the generators of $\Gamma(4,4,4;5)$ either multiplying them together and checking they satisfy the required group relations or by comparing the matrices to the generators I_i in either [6] or [21]. In these coordinates the polar vectors for $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$ and $\mathcal{C}_{121}$ are:

$$\mathbf{n}_1 = \begin{bmatrix} -\phi\overline{\omega} \\ -\phi^2 \\ \phi\sqrt{\phi} \end{bmatrix}, \quad \mathbf{n}_2 = \begin{bmatrix} \frac{\phi(\phi-\overline{\omega})}{2} \\ -\frac{\phi(\phi-\overline{\omega})}{2} \\ \phi\sqrt{\phi} \end{bmatrix}, \quad \mathbf{n}_3 = \begin{bmatrix} 1-\phi\overline{\omega} \\ -1 \\ \phi\sqrt{\phi} \end{bmatrix}, \quad \mathbf{n}_{121} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

Lemma 4.1.3 $\mathbb{A}(\boldsymbol{n}_1, \boldsymbol{n}_2, \boldsymbol{n}_3) = \arg(-\phi^2(\phi^4\omega - 1)/2) = 2.211616\ldots$. *So* $t = \pi - \mathbb{A}(\boldsymbol{n}_1, \boldsymbol{n}_2, \boldsymbol{n}_3) = 0.929976\ldots$.

Proof: Recall $\mathbb{A}(\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3) = \arg\left(-\langle \mathbf{n}_3, \mathbf{n}_2\rangle\langle \mathbf{n}_2, \mathbf{n}_1\rangle\langle \mathbf{n}_1, \mathbf{n}_3\rangle\right)$, then the result follows from a straightforward calculation. Notice that $\mathbb{A}(\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3)$ agrees with Deraux's t (equation (2.8) of [6]). □

Using the formulae from chapter 2 we calculate the values of t_0 and t_1 for deformed $(4,4,4)$ groups, as $t_0 = 1.209429\ldots$ and $t_1 = 0.785398\ldots$. Since the value of t corresponding to $\Gamma(4,4,4;5)$ lies between t_0 and t_1, we confirm the group $\Gamma(4,4,4;5)$ lies in the region of the parameter space corresponding to non-degenerate triangle groups where I_{1323} is regular elliptic
We also define the following matrices

$$P = \begin{pmatrix} e^{-\pi i/9}\left(\omega - \phi\right) & 0 & e^{-\pi i/9}\left(\phi\sqrt{\phi}\right) \\ 0 & e^{2\pi i/9}\overline{\omega} & 0 \\ e^{-\pi i/9}\left(\omega\phi\sqrt{\phi}\right) & 0 & e^{-\pi i/9}\left(1 - \omega\phi\right) \end{pmatrix},$$

$$J = \begin{pmatrix} -\omega\phi^2 & \omega\phi^2 & \phi\sqrt{\phi}\,(\overline{\omega} + 2) \\ -\overline{\omega}\phi & -\omega\phi & -w\phi\sqrt{\phi} \\ \overline{\omega}\phi\sqrt{\phi} - \omega\sqrt{\phi} & -\phi\sqrt{\phi}\,(1 + \overline{\omega}\phi) & \omega\phi^3 \end{pmatrix},$$

$$P^5 = \begin{pmatrix} e^{\pi i/9} & 0 & 0 \\ 0 & e^{-2\pi i/9} & 0 \\ 0 & 0 & e^{\pi i/9} \end{pmatrix},$$

$$J' = I_1 J I_1 = e^{-\pi i/9}\begin{pmatrix} \phi^2 & -\phi & (\overline{\omega} - \phi)\sqrt{\phi} \\ -\phi & -\omega\phi & \phi\sqrt{\phi} \\ (1 - \omega\phi)\sqrt{\phi} & \omega\phi\sqrt{\phi} & \omega\phi - \phi^2 \end{pmatrix}.$$

The matrix J is an order 3 regular elliptic isometry that permutes $\mathbf{n}_1$, $\mathbf{n}_2$ and $\mathbf{n}_3$. The matrix J' cyclically permutes $\mathbf{n}_{131}$, $\mathbf{n}_1$ and $\mathbf{n}_{121}$.

Rather than show that $\Gamma(4,4,4;5)$ is a lattice we instead prove that $\langle I_1, J\rangle$ is a lattice and since $\Gamma(4,4,4;5)$ is an index three normal subgroup of $\langle I_1, J\rangle$, it follows that $\Gamma(4,4,4;5)$ is a lattice with three times the covolume of $\langle I_1, J\rangle$.

Lemma 4.1.4 *P has order 30 and P^5 is a complex reflection of order 6.*

PROOF: This can easily be checked via a computer calculation or calculating eigenvalues of the matrices above. □

Lemma 4.1.5 *$P = JI_1$ and $P^3 = I_{231}$.*

PROOF: The first relation can be quickly confirmed using Maple. For the second relation we observe $JI_iJ^{-1} = I_{i+1}$ taking the indices modulo 3. We use this relation to simplify $P^3 = JI_1JI_1JI_1 = I_{231}$. □

Proposition 4.1.6 *Deraux's lattice is arithmetic, in particular, it is discrete.*

PROOF: Corollary 2.6 of [6]. □

4.1.2 The action of P

Since P will play an important role in our construction, for convenience we now include the relevant part of the P-orbit of words which will appear as either vertices or $\mathbb{C}$-lines in our fundamental domain. In word notation, P, like all isometries, acts on a word by conjugation. The calculation can be easily checked e.g. $P(1231) = J112311J^{-1} = J23J^{-1} = 31$.

Words corresponding to points

$$1231 \to 13 \to 12 \to 23 \to 2312,$$
$$12321231 \to 132313 \to 131212 \to 1323 \to 2131 \to 1232 \to 2313 \to 232121,$$
$$2321232121 \to 2313231323 \to 231213121312 \to 32312321 \to 21312313 \to 2321232121.$$

Words corresponding to $\mathbb{C}$-lines

$$131 \to 1 \to 2 \to 232,$$

$$313 \to 121 \to 3 \to 212 \to 323.$$

Remark: These orbits show a degree of symmetry with respect to the anti-holomorphic involution that fixes 3 and swaps 1 and 2. On the first orbit this involution fixes 12 (also $P^{15}(12)$) and swaps $P^n(12)$ with $P^{-n}(12)$ for $-15 < n < 15$. On the second orbit, the involution swaps $P^n(1232)$ with $P^{-(n+1)}(1232)$. Similar symmetries exist on the other three orbits (compare with section 4.2 of [30]).

Proposition 4.1.7 *The group $\langle I_{132}, I_{2321232121}\rangle$ stabilises a $\mathbb{C}$-line and acts on that $\mathbb{C}$-line as an index 2 subgroup of a $\mathbb{C}$-Fuchsian $(2,5,6)$ triangle group.*
The group $\langle I_{121}, I_2, I_3\rangle$ stabilises an $\mathbb{R}$-plane and acts on that $\mathbb{R}$-plane as an $\mathbb{R}$-Fuchsian $(2,4,5)$-reflection group.

Proof: The matrices for I_{132} and $I_{2321232121}$ are

$$I_{132} = \begin{pmatrix} \omega\phi & 0 & \sqrt{\phi} \\ 0 & \overline{\omega} & 0 \\ -\omega\sqrt{\phi} & 0 & -\phi \end{pmatrix} \quad I_{2321232121} = \begin{pmatrix} -\overline{\omega} & 0 & 0 \\ 0 & -\omega & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Both words preserve the $\mathbb{C}$-line $\{(\zeta,0,1)^t | \zeta \in \mathbb{C})\}$. After checking eigenvalues, we see that I_{132} acts on this $\mathbb{C}$-line as an order 5 rotation about the point $*$ (see section 4.3.1) and $I_{2321232121}$ acts as an order 6 rotation about $(0,0,1)^t$ and $I_{132}I_{2321232121} = I_{121232121}$ is and order 2 isometry also fixing the same $\mathbb{C}$-line. A $\mathbb{C}$-line is a copy of the $\mathbf{H}^2_{\mathbb{R}}$, so we conclude that $\langle I_{132}, I_{2321232121}\rangle$ is an index 2-subgroup of a $(2,5,6)$ triangle group.

The second part is essentially identical to lemma 3.2 from [31]. □

4.2 A combinatorial fundamental polyhedron

In this section we give a combinatorial description of a polyhedron that will later be modified to produce a fundamental domain for $\langle I_1, J\rangle$.

Our fundamental domain for the group will consist of three 'core' polyhedra (all contained in bisectors) over which we take a geodesic cone (a process described in section 4.2.2). This process is strongly related to how we constructed the fundamental domains for $\Gamma(3,3,4;7)$ and $\Gamma(3,3,5;5)$ described in chapter 3, however we are able to simplify the construction in this case by using the extra symmetry J to reduce the number of core faces required.

4.2.1 The core faces

There are three core pieces, these are solid polyhedra homeomorphic to the 3 dimensional ball. We denote them **A**, **B** and **C** (as shown in figure 4.1).

The first core piece, **A**, is the octahedron with vertices $1231, 12, 13, 1323, 1232, 23$. Note that $12, 13, 1323, 1232$ all lie in the $\mathbb{C}$-plane $\mathcal{C}_1$ and $I_1(1232) = 23$. The complex reflection I_1 in $\mathcal{C}_1$ sends **A** to itself. This is essentially the same as Schwartz's Odd B pieces in section 4.6 [30].

The second core piece, **B**, is the solid affine decahedron (or more accurately pentagonal dipyramid) with vertices 1232, 232121, 12, 1231, 12321231, 2321232121, 131212. The points 232121, 12, 1231, 12321231, 2321232121 all lie in the $\mathbb{C}$-plane $\mathcal{C}_{121}$ and $I_{121}\,(1232) = 131212$. This is equivalent to the five sided version of Schwartz's even A Pieces).

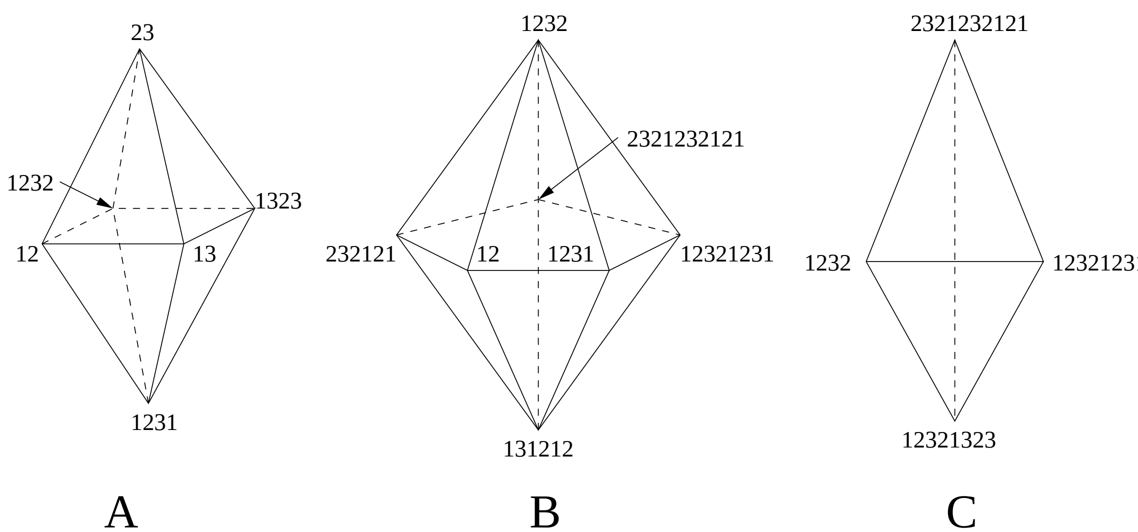

Figure 4.1: The core faces **A**, **B** and **C**

The final core face **C** is a tetrahedron with vertices 2321232121, 1232, 12321231 and 12321323. Note that 2321232121 and 12321323 are fixed by P^5 and $P^5(12321231) = 1232$. Also $I_{123212321}$ sends **C** to itself.

By identifying any triangular faces that have the same vertices we glue our three core pieces together as follows; we glue **A** and **B** along $(1232, 12, 1231)$ and glue **B** and **C** along
$(2321232121, 123212321, 1232)$. **A** and **C** do not share any common faces.

Definition 4.2.1 *We call the polyhedron* $\boldsymbol{A} \cup \boldsymbol{B} \cup \boldsymbol{C}$ *with the above gluing* ***the core polyhedron*** *and* $\boldsymbol{A}$, $\boldsymbol{B}$ *and* $\boldsymbol{C}$ ***core faces****.*

Proposition 4.2.2 *Images of the core polyhedra under* $\langle P \rangle$ *tile a region homeomorphic to the 3-sphere, in particular all the 2-faces 'close up', by which we mean every 2-face is contained in exactly two polyhedra of the* P*-orbit (this is equivalent to the* P*-orbit of the core polyhedra forming a 4-dimensional polytope).*

PROOF: We first consider the P-orbit of **A**, the core octahedron. **A** shares common 2-faces with the following elements of its P-orbit

$$\mathbf{A} \cap P(\mathbf{A}) = (13, 12, 23),$$
$$\mathbf{A} \cap P^{-1}(\mathbf{A}) = (1231, 13, 13),$$
$$\mathbf{A} \cap P^{2}(\mathbf{A}) = (12, 23, 1232),$$
$$\mathbf{A} \cap P^{-2}(\mathbf{A}) = (1231, 13, 1323).$$

This can be difficult to visualize, but is readily seen with the aid of a model.

We can continue with this process for the whole P-orbit gluing $P^i(\mathbf{A})$ to $P^{i+k}(\mathbf{A})$ where $k \in \{-2, -1, 1, 2\}$ for all $0 \le i \le 29$.

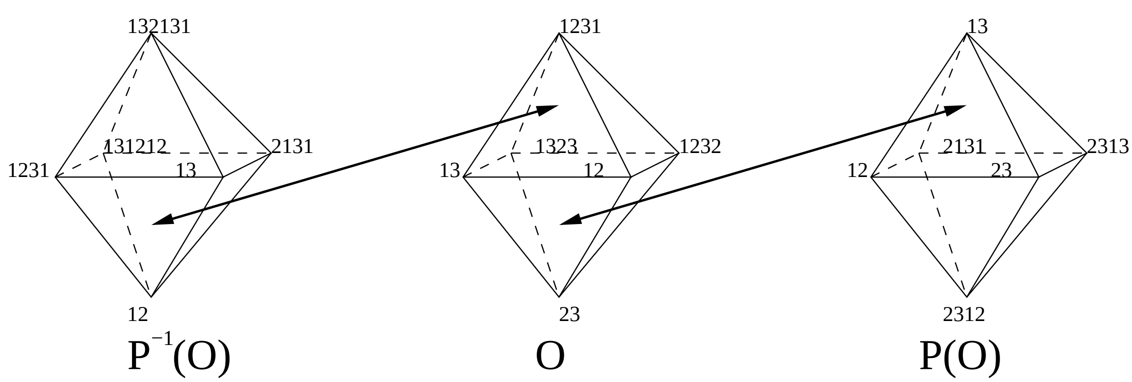

Figure 4.2: **A**, $P^{-1}(\mathbf{A})$, and $P(\mathbf{A})$.

Since P has order 30 this process will eventually terminate as the two ends of the tower of octahedra will be identified forming a polyhedron homeomorphic to a solid torus. The surface of this torus is triangulated by the faces of $\{P^i(\mathbf{A})\}_{i=0}^{29}$ that don't glue to any other octahedron in the orbit. This configuration is shown in figure 4.3, the numbers inside triangles denote the power of P to which that face belongs. This is locally identical to figure 6.2 of [30] except, in that case, the 'tower' of octahedra does not close up to form a torus.

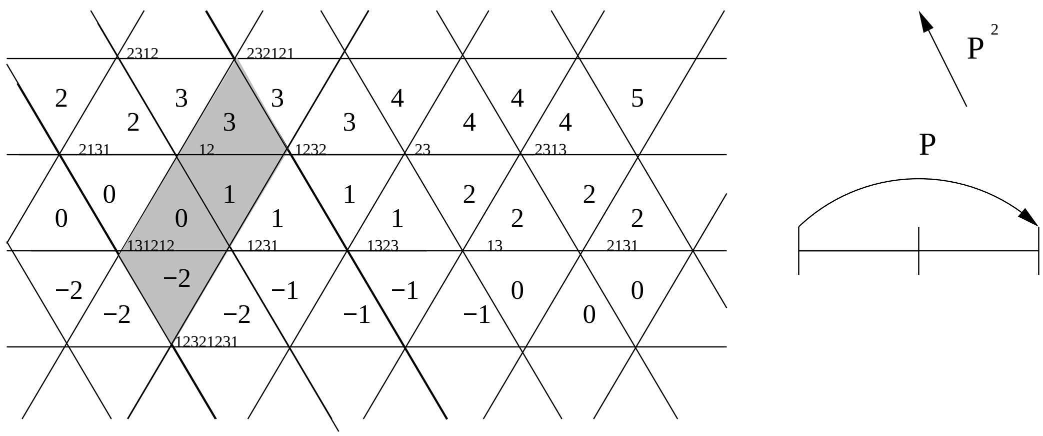

Figure 4.3: Outside of the torus of octahedra.

Onto the surface of the torus we glue the P-orbit of the decahedron $\mathbf{B}$, (this is analogous to adding the Z_2 layer in Schwartz), the shaded regions on figures 4.3 and 4.4 are the attaching site of $\mathbf{B}$ and the whole surface is covered by translates of this region under P, i.e. all the faces of the octahedra are glued to a face of another polyhedron.

Now the faces of the decahedron are glued to the faces belonging to other decahedra as follows,

$$\begin{aligned}
P^2(\mathbf{B}) \cap \mathbf{B} &= (232121, 12, 13121), \\
P^{-2}(\mathbf{B}) \cap \mathbf{B} &= (1232, 1231, 12321231), \\
P^5(\mathbf{B}) \cap \mathbf{B} &= (2321232121, 232121, 1232), \\
P^{-5}(\mathbf{B}) \cap \mathbf{B} &= (2321232121, 131212, 12321231).
\end{aligned}$$

At this stage, all the faces of the octahedra $P^n(\mathbf{A})$ are closed and all except two of the faces on each decahedron $P^n(\mathbf{B})$ are closed. The remaining sixty faces are

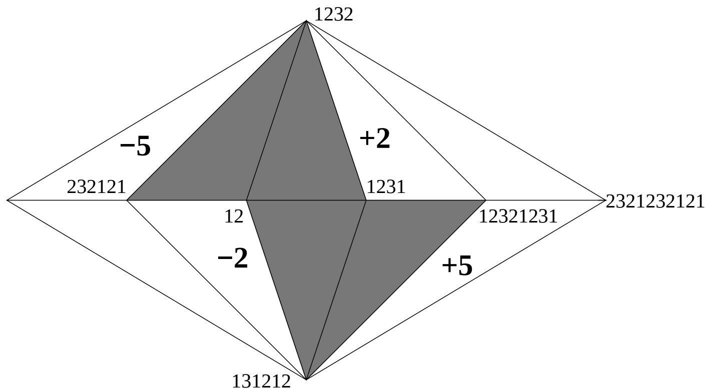

Figure 4.4: Schematic figure of the gluing of **B**

on P-orbit of **B** are $(2321232121, 1232, 12321231)$ and $(2321232121, 131212, 232121)$ and their images under P. We glue onto these faces the P-orbit of **C** as follows.

$$\mathbf{B} \cap \mathbf{C} = (2321232121, 1232, 12321231)\,,$$
$$\mathbf{B} \cap P^2\,(\mathbf{C}) = (2321232121, 131212, 232121)\,.$$

This only leaves two faces of **C** not closed (and their images under P). These faces are

$(2321232121, 12321323, 1232)$ and $(2321232121, 12321323, 12321231)$. They are glued to neighbouring copies of **C** as follows

$$\mathbf{C} \cap P^5\,(\mathbf{C}) = (2321232121, 12321323, 1232)\,,$$
$$\mathbf{C} \cap P^{-5}\,(\mathbf{C}) = (2321232121, 12321323, 12321231)\,.$$

Since every 2-dimensional face in the P-orbit of $(\mathbf{A} \cap \mathbf{B} \cap \mathbf{C})$ is contained in exactly two polyhedra we conclude that it is polytope and it is homeomorphic to a 3-sphere. □

4.2.2 The cone faces

The full combinatorial fundamental domain which we denote Δ is the geodesic cone over the core polyhedron $(\mathbf{A} \cup \mathbf{B} \cup \mathbf{C})$ to the fixed point of P, which we denote $*$. Since we are only concerned with affine structure in this section, this is a purely combinatorial construction. In the next section we need to take more care ensuring the cones have no self intersections. For a subset $D \subset \mathbf{H}^2_{\mathbb{C}}$ and a point $p \in \mathbf{H}^2_{\mathbb{C}}$ we define $\mathrm{Cone}_p\left(D\right)$ as follows:

$$\mathrm{Cone}_p\left(D\right) = \bigcup_{z \in D} [z,p] \tag{4.6}$$

where $[z,p]$ is the geodesic segment from z to p.

Definition 4.2.3 *Let $*$ be the fixed point of P (equivalently the fixed point of $P^3 = I_{132}$). Our fundamental domain Δ is defined as*

$$\Delta := \left(\mathrm{Cone}_*\left(\boldsymbol{A} \cup \boldsymbol{B} \cup \boldsymbol{C}\right)\right) \cup \left(\boldsymbol{A} \cup \boldsymbol{B} \cup \boldsymbol{C}\right). \tag{4.7}$$

The interior of Δ is the cone over the interior of the core polyhedron $\boldsymbol{A} \cup \boldsymbol{B} \cup \boldsymbol{C}$ and the boundary of Δ is the prism over the boundary of $\boldsymbol{A} \cup \boldsymbol{B} \cup \boldsymbol{C}$ (i.e. the cone over the 2-faces in the boundary of the core pieces constructed above) together with the core faces themselves.

The boundary of Δ, consists of twenty one 3-faces, namely $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and eighteen cone faces, P_i, as listed below. Later, when we use Poincaré's polyhedron theorem, will simplify this configuration by combining several of the cone faces together and reducing the total number of 3-faces to eleven.

List of cone faces

The cones that appear as faces in the fundamental domain are the cones over the exposed faces in the triangulated surface of the core polyhedron $\mathbf{A} \cup \mathbf{B} \cup \mathbf{C}$,

$$\begin{gathered}
\mathrm{Cone}_*(13,12,23),\ \mathrm{Cone}_*(13,23,1323),\ \mathrm{Cone}_*(1231,13,12),\ \mathrm{Cone}_*(1231,12,131212),\\
\mathrm{Cone}_*(23,1323,1232),\ \mathrm{Cone}_*(1231,12321231,131212),\ \mathrm{Cone}_*(12,23,1232),\\
\mathrm{Cone}_*(12,131212,232121),\ \mathrm{Cone}_*(12,1232,232121),\ \mathrm{Cone}_*(131212,232121,2321232121),\\
\mathrm{Cone}_*(12321231,1232,12321323),\ \mathrm{Cone}_*(1231,12321231,1232),\ \mathrm{Cone}_*(1231,1323,1232),\\
\mathrm{Cone}_*(1231,13,1323),\ \mathrm{Cone}_*(1232,232121,2321232121),\ \mathrm{Cone}_*(1232,12321323,2321232121),\\
\mathrm{Cone}_*(12321231,131212,2321232121),\ \mathrm{Cone}_*(12321231,12321323,2321232121).
\end{gathered}$$

4.3 A fundamental domain

We now construct a true fundamental domain for Deraux's lattice by modifying the combinatorial polyhedron described in the previous section (this process is similar to [9]). This is done by keeping the same 0-skeleton, replacing the 1-skeleton with geodesic segments and the 2-skeleton with the regions of the 2-dimensional intersection of bisectors bounded by the 1-skeleton. The core faces in the 3-skeleton are replaced with the regions of bisectors bounded by the 2-skeleton. The cone faces are replaced with the geodesic cones over the 2-skeleton of the core faces. The cone faces will not be contained in any bisectors. We call this modified polytope the **geometric fundamental domain** (to contrast with the combinatorial fundamental domain) and the modified core and cone faces, **geometric faces**.

4.3.1 The vertices of Δ

Using the matrices described in section 4.1.1 we can explicitly calculate the coordinates of the vertices that appear as the 0-skeleton of our fundamental domain. The vector $*$ corresponds to the unique fixed point of P and

$$\Upsilon := (1/2)\phi^2\left(1-\overline{\omega}+\omega\sqrt{4\phi-7}\right).$$

$$1232 = \begin{bmatrix} -\overline{\omega} \\ -1 \\ \phi\sqrt{\phi} \end{bmatrix} \quad 2321232121 = \begin{bmatrix} 0 \\ 0 \\ \phi\sqrt{\phi} \end{bmatrix} \quad 232121 = \begin{bmatrix} 1 \\ -1 \\ \phi\sqrt{\phi} \end{bmatrix}$$

$$12 = \begin{bmatrix} \frac{\phi-\overline{\omega}}{2} \\ \frac{\overline{\omega}-\phi}{2} \\ \phi\sqrt{\phi} \end{bmatrix} \quad 1231 = \begin{bmatrix} \frac{1-\phi\overline{\omega}}{2} \\ \frac{\phi\overline{\omega}-1}{2} \\ \phi\sqrt{\phi} \end{bmatrix} \quad 12321231 = \begin{bmatrix} -\overline{\omega} \\ \overline{\omega} \\ \phi\sqrt{\phi} \end{bmatrix}$$

$$131212 = \begin{bmatrix} 1 \\ \overline{\omega} \\ \phi\sqrt{\phi} \end{bmatrix} \quad 13 = \begin{bmatrix} \frac{1}{3}\phi^2\left(1-\overline{\omega}\right) \\ \frac{1}{3}\phi\left(\overline{\omega}-1\right) \\ \phi\sqrt{\phi} \end{bmatrix} \quad 1323 = \begin{bmatrix} \frac{1}{2}\left(1-\overline{\omega}\phi^2\right) \\ \frac{1}{2}\left(\overline{\omega}\phi+\omega\right) \\ \phi\sqrt{\phi} \end{bmatrix}$$

$$23 = \begin{bmatrix} \omega+\phi \\ -1 \\ \phi\sqrt{\phi} \end{bmatrix} \quad 32312321 = \begin{bmatrix} -\overline{\omega}\phi \\ 0 \\ \phi\sqrt{\phi} \end{bmatrix} \quad * = \begin{bmatrix} \Upsilon \\ 0 \\ \phi\sqrt{\phi} \end{bmatrix}.$$

Remark: We have chosen to renormalise the standard lift of a point by multiplying by $\phi\sqrt{\phi}$, this is merely for convenience since the coordinates can be more naturally expressed in this form.

4.3.2 The core faces

We now construct three bisectors, that contain the vertices of each of the core faces and then using the totally geodesic decompositions of bisectors (proposition 1.2.25) we shall put geodesic structure on the 1-skeleton of the core faces. Recall that a bisector can be uniquely determined by its real spine (a real geodesic). Let σ_B be the real geodesic $\gamma(131212, 1232)$ contained in the $\mathbb{C}$-line $\Sigma(131212, 1232)$. Then define $\mathcal{B}_B$ as the bisector determined by this geodesic.

Lemma 4.3.1 *$\mathcal{C}_{121}$ is a slice of $\mathcal{B}_B$ and any geodesic between a pair of points from the following set lies entirely within $\mathcal{B}_B$*

$$\{p_{1232}, p_{232121}, p_{12}, p_{1231}, p_{12321231}, p_{2321232121}, p_{131212}\}$$

Proof: First observe that I_{121}, the inversion in $\mathcal{C}_{121}$ interchanges 1232 and 131212 and hence σ_{121}, the geodesic between these points, is fixed set-wise by I_{121}. Since a bisector is uniquely determined by its real spine $\mathcal{B}_B = I_{121}(\mathcal{B}_B)$ then by 5.2.1 of [10] $\mathcal{C}_{121}$ is a slice of $\mathcal{B}_B$. The second part follows from the observation that all the points from the list lie in either the slice $\mathcal{C}_{121}$ or on the real spine and so any pair of points lie in a totally geodesic subspace (either a slice or a meridian). $\square$

Corollary 4.3.2 *By an essentially identical argument we can find bisectors $\mathcal{B}_A$ and $\mathcal{B}_C$ containing our half-octahedron and tetrahedron faces **A** and **C**. For $\mathcal{B}_A$ the real spine σ_A is the real geodesic containing 23 and 1232 in the $\mathbb{C}$-line $\Sigma(23, 1231)$. For $\mathcal{B}_C$ the real spine σ_C is the real geodesic containing the points 2321232121 and 12321323 in the $\mathbb{C}$-line $\Sigma(2321232121, 12321323)$.*

We put a geodesic structure on their respective 1-skeletons in exactly the same manner.

Bisectors containing core faces

$$\begin{aligned}
\mathcal{B}_A &= \mathcal{E}\left(\begin{pmatrix} \omega\phi \\ -\omega+\overline{\omega} \\ -\omega\sqrt{\phi}+(\omega-\overline{\omega})\phi\sqrt{\phi} \end{pmatrix}, \begin{pmatrix} 1-\omega-\phi \\ 0 \\ \overline{\omega}\sqrt{\phi} \end{pmatrix}\right) \\
&= \left\{\begin{pmatrix} z_1 \\ z_2 \\ \phi\sqrt{\phi} \end{pmatrix} \;:\; \left|\overline{\omega}\phi z_1+(\omega-\overline{\omega})z_2+\overline{\omega}\phi^2+(\omega-\overline{\omega})\phi^3\right| = \left|\overline{\omega}\phi z_1+(\omega-\overline{\omega})z_1+\phi^2\right|\right\} \\
&= \left\{\begin{pmatrix} z \\ (z+\phi-e^{i\theta}z-e^{i\theta}\omega\phi)(1-\omega)\phi/3-\phi^3+e^{i\theta}z \\ \phi\sqrt{\phi} \end{pmatrix} \;:\; z\in\mathbb{C},\ \theta\in[0,2\pi)\right\},
\end{aligned}$$

$$\begin{aligned}
\mathcal{B}_B &= \mathcal{E}\left(\begin{pmatrix} 0 \\ \overline{\omega}-1 \\ \phi\sqrt{\phi} \end{pmatrix}, \begin{pmatrix} 1-\overline{\omega} \\ 0 \\ \phi\sqrt{\phi} \end{pmatrix}\right) \\
&= \left\{\begin{pmatrix} z_1 \\ z_2 \\ \phi\sqrt{\phi} \end{pmatrix} \;:\; 3\left(|z_2|^2-|z_1|^2\right)+2\phi^3\mathrm{Re}\left((1-\omega)\left(z_1+z_2\right)\right)=0\right\} \\
&= \left\{\begin{pmatrix} z \\ ze^{i\theta}-\phi^3(1+e^{i\theta})(1-\overline{\omega})/3 \\ \phi\sqrt{\phi} \end{pmatrix} \;:\; z\in\mathbb{C},\ \theta\in[0,2\pi)\right\},
\end{aligned}$$

$$\begin{aligned}
\mathcal{B}_C &= \mathcal{E}\left(\begin{pmatrix} 1-\omega \\ 0 \\ \phi\sqrt{\phi} \end{pmatrix}, \begin{pmatrix} \omega-1 \\ 0 \\ \phi\sqrt{\phi} \end{pmatrix}\right) \\
&= \left\{\begin{pmatrix} z_1 \\ z_2 \\ \phi\sqrt{\phi} \end{pmatrix} \;:\; \mathrm{Re}\left(z_1\left(1-\overline{\omega}\right)\right)=0\right\} \\
&= \left\{\begin{pmatrix} t\,\overline{\omega} \\ z \\ \phi\sqrt{\phi} \end{pmatrix} \;:\; t\in\mathbb{R},\ z\in\mathbb{C}\right\}.
\end{aligned}$$

The polar vectors to the complex spines are as follows,

$$\mathbf{n}_A = \begin{pmatrix} \frac{\phi^2}{2}(\phi - \overline{\omega}) \\ \frac{\overline{\omega}\phi - \omega}{2} \\ \phi\sqrt{\phi} \end{pmatrix}, \; \mathbf{n}_B = \begin{pmatrix} \frac{\phi^3(1-\overline{\omega})}{3} \\ -\frac{\phi^3(1-\overline{\omega})}{3} \\ \phi\sqrt{\phi} \end{pmatrix}, \; \mathbf{n}_C = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

The real spines are the intersections of the complex spines with the bisectors, explicitly

$$\sigma_A = \begin{pmatrix} (\phi^2 - \omega)\sinh((t-s)/2) - (\phi\overline{\omega} - 1)\sinh((r-t)/2) \\ (\omega - \phi)\sinh((t-s)/2) + (\phi\overline{\omega} - 1)\sinh((r-t)/2) \\ \phi\sqrt{\phi}\,((\phi - \omega)\sinh((t-s)/2) + 2\sinh((r-t)/2)) \end{pmatrix},$$

$$\sigma_B = \begin{pmatrix} \omega t - \overline{\omega} \\ \omega t - 1 \\ \phi\sqrt{\phi} \end{pmatrix}, \; \sigma_C = \begin{pmatrix} t\overline{\omega} \\ 0 \\ \phi\sqrt{\phi} \end{pmatrix}.$$

(unfortunately σ_A does not seem to have a nice form in these coordinates).

Bisector intersections containing 2-faces

This section contains a list of the pairs of bisectors whose intersections contain one of the exposed 2-faces of the core pieces. For example the face $(1232, 2321232121, 232121)$ is contained in both $\mathbf{B}$ and $P^5(\mathbf{B})$ and so the geometric face is contained in the intersection of $\mathcal{B}_B$ (the bisector containing $\mathbf{B}$) and $P^5(\mathcal{B}_B)$ (the bisector containing $P^5(\mathbf{B})$). These intersections are Giraud discs (see 1.2.32) so there is at most one other bisector containing the intersection.

Using these bisectors and their images under various elements of our group we can now explicitly write down the 2-dimensional submanifolds that will contain the generic 2-faces of our core pieces, it will be a smooth disc containing three geodesic segments. The 2-faces will be the regions of these discs bounded by the geodesic 1-skeleton. The core 3-faces will then simply be the regions of the bisector bounded by the 2-faces.

2-faces of $\mathbf{A}$

- $(1231, 13, 12) \subset \mathcal{B}_A \cap P^{-1}\mathcal{B}_A$,

- $(1231, 13, 1323) \subset \mathcal{B}_A \cap P^{-2}\mathcal{B}_A$,
- $(1231, 1323, 1232) \subset \mathcal{B}_A \cap P^{-2}\mathcal{B}_B$,
- $(12, 23, 1232) \subset \mathcal{B}_A \cap P^{2}\mathcal{B}_A$,
- $(13, 12, 23) \subset \mathcal{B}_A \cap P\mathcal{B}_A$,
- $(13, 23, 1323) \subset \mathcal{B}_A \cap P\mathcal{B}_B$,
- $(23, 1323, 1232) \subset \mathcal{B}_A \cap P^{3}\mathcal{B}_B$.

2-faces of B

- $(1232, 232121, 2321232121) \subset \mathcal{B}_B \cap P^{5}\mathcal{B}_B$,
- $(12, 1232, 232121) \subset \mathcal{B}_B \cap P^{2}\mathcal{B}_A$,
- $(1231, 12321231, 1232) \subset \mathcal{B}_B \cap P^{-2}\mathcal{B}_B$,
- $(131212, 232121, 2321232121) \subset \mathcal{B}_B \cap P^{2}\mathcal{B}_C$,
- $(12, 131212, 232121) \subset \mathcal{B}_B \cap P^{2}\mathcal{B}_B$,
- $(1231, 12, 131212) \subset \mathcal{B}_B \cap P^{-1}\mathcal{B}_A$,
- $(1231, 12321231, 131212) \subset \mathcal{B}_B \cap P^{-3}\mathcal{B}_A$,
- $(12321231, 131212, 2321232121) \subset \mathcal{B}_B \cap P^{-5}\mathcal{B}_B$.

2-faces of C

- $(12321231, 1232, 12321323) \subset \mathcal{B}_C \cap P^{-2}\mathcal{B}_B$,
- $(12321231, 12321323, 2321232121)$ is contained in a meridian of $\mathcal{B}_C$,
- $(1232, 12321323, 2321232121)$ is contained in a meridian of $\mathcal{B}_C$.

We will take geodesic cones over these eighteen 2-faces. In addition there are also the two 2-faces along which we glued the core pieces, we list these 2-faces for the sake of completeness.

- $(1231, 12, 1232) \subset \mathcal{B}_A \cap \mathcal{B}_B$,

- $(12321231, 1232, 2321232121) \subset \mathcal{B}_B \cap \mathcal{B}_C$.

Proposition 4.3.3 *The intersection of any pair of bisectors in the above list will be a smooth 2-dimensional disc.*

PROOF: This immediately follows from the fact that the complex spines of the bisectors intersect outside their real spines. This is a straightforward calculation on Maple using the polar vectors above and the observation that the complex spines only meet at one point, so we need only ensure that this intersection point is not contained in either real spine. All the above pairs of bisectors are coequidistant and therefore by 1.2.32 the intersection is a smooth connected disc. □

Definition 4.3.4 *A bisector $\mathcal{B}$ partitions $\boldsymbol{H}^2_{\mathbb{C}} \setminus \mathcal{B}$ into two disjoint half-spaces, if $*$ is not contained in $\mathcal{B}$, then we say the half-space containing $*$ is the* ***good side*** *of $\mathcal{B}$ (with respect to $*$). The other half space we call the* ***bad side*** *of $\mathcal{B}$.*

Lemma 4.3.5 *For $n \in \{0, 1, \ldots 29\}$, $*$ does not lie in $P^n(\mathcal{B}_A)$, $P^n(\mathcal{B}_B)$ or $P^n(\mathcal{B}_C)$. In other words all these bisectors have a well defined good side.*

PROOF: This follows from the observations that $*$ is not contained in $\mathcal{B}_A$, $\mathcal{B}_B$ or $\mathcal{B}_C$ and $*$ is fixed under P. □

Lemma 4.3.6 *For $n \in \{0, 1, \ldots 29\}$ the points $P^n(12)$, $P^n(1232)$ and $P^n(232123212)$ are either contained in $\mathcal{B}_A$ or they are on the good side of $\mathcal{B}_A$ (the same half space as $*$).*

PROOF: Since we have already explicitly calculated the points 12, 1232 and 232123212 and the matrix P, this can be checked relatively quickly using Maple. The same is true for $\mathcal{B}_B$ and $\mathcal{B}_C$. □

Lemma 4.3.7 *For each of the bisectors $\mathcal{B}_A$, $\mathcal{B}_B$ or $\mathcal{B}_C$, the P-orbit of the geodesic segments that make up the 1-skeleton of the core faces is contained in the bisector or on its good side.*

Proof: Again, this can be checked using Maple, although it is rather a lengthy and tedious series of calculations. □

Lemma 4.3.8 *The geodesic segment* $[p, *]$*, for* $p \in \{P^n(12), P^n(1232), P^n(232123212) \mid n = 0, 1, \ldots 29\}$*, is contained in the good sides of* $\mathcal{B}_A$*,* $\mathcal{B}_B$ *and* $\mathcal{B}_C$*.*

The four lemmas above together show that the 0 and 1-skeletons of Δ lie either in or on the good side of $P^n(\mathcal{B}_A)$, $P^n(\mathcal{B}_B)$ and $P^n(\mathcal{B}_C)$ for $n \in \{0, 1, \ldots 29\}$. Equivalently, the 1-skeleton of $P^n(\Delta)$ lies on the good side of $\mathcal{B}_A$,$\mathcal{B}_B$ and $\mathcal{B}_C$. Eventually we will show that all of Δ is contained on the good sides of these bisectors.

4.3.3 Intersection of faces

This section can be skipped over by a reader who is prepared to believe that the geometric and combinatorial fundamental domains are homeomorphic, i.e. the geometric faces do not intersect anywhere not prescribed by the combinatorial domain.

Theorem 4.3.9 *The polyhedron* Δ *is topologically a four dimensional ball and the boundary* $\partial\Delta$ *is a three dimensional sphere.*

To prove this we use the following theorem.

Theorem 4.3.10 *Let* p *be a point on the boundary of the core polyhedron, then* $[p, *]$ *the geodesic line segment between* p *and* $*$*, does not intersect* $\boldsymbol{A}$*,* $\boldsymbol{B}$ *or* $\boldsymbol{C}$ *except where specified by the combinatorial model.*

This is a technical result which leads to the following corollaries.

Corollary 4.3.11 *Cone faces are homeomorphic to solid 3-balls.*

Proof: The only way a cone face could fail to be homeomorphic to a solid 3-ball is if there were two points, p and p', in the triangle base with the property that $[p, *] \subset [p', *]$. In other words the point p lies on the geodesic between p' and $*$. But since p is contained in one of the core faces, this would mean that the geodesic segment $[p', *]$ passes through a core face at a point not prescribed by the combinatorial model. By theorem 4.3.10 this does not occur. □

Corollary 4.3.12 *Two cone faces intersect only where specified by the combinatorial model.*

PROOF: This is essentially the same argument as the proof of the previous corollary, if the cone faces did intersect somewhere that the combinatorial model does not specify, that would necessarily mean there are two geodesics one of which is a subset of the other, which leads to the same contradiction as before, i.e. one of the cone faces would have a bad intersection with a core face. □

Corollary 4.3.13 *A cone face and a core face only intersect as in the combinatorial model.*

PROOF: This is literally what theorem 4.3.10 says. □

Theorem 4.3.10 would be simple to prove if we could assume that if two points p and q lie on the same side of a bisector, then the entire geodesic segment $[p, q]$ also lies on that side of the bisector. However this would imply that bisectors are totally geodesic, which unfortunately is not the case.
There is also the possibility that the cones over the interior of the core pieces intersect badly. Concretely there could, for example, be points $p \in \mathrm{int}(\mathbf{A})$ and $q \in \mathrm{int}(\mathbf{B})$ such that $[p, *] \subset [q, *]$, see the right hand configuration in figure 4.5 for a schematic picture. This cannot happen since it would necessarily require part of the 1-skeleton

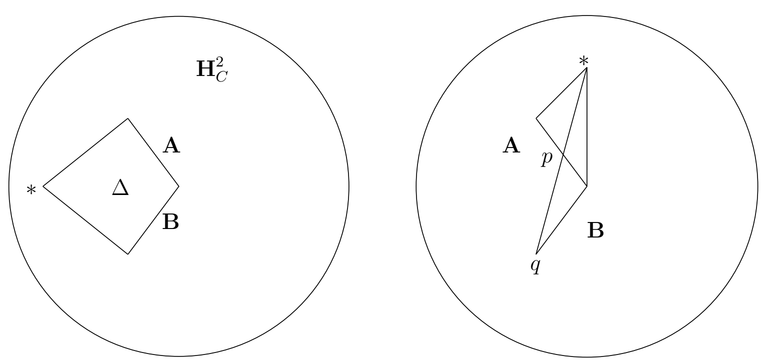

Figure 4.5: Left hand side: Simplified schematic view of Δ. Right hand side :Schematic of a possible bad intersection

of Δ to lie on the bad side of the bisector containing one of the core faces. In the schematic picture the part of the one skeleton is on the bad side of $\mathcal{B}_A$, the bisector containing **A**. Since we have established that the 1-skeleton is entirely contained in the good sides of all three bisectors we conclude this sort of intersection is impossible.

We also need to ensure that no two core faces intersect outside of the prescribed 2-faces. This follows directly from the fact that we constructed our core faces to be regions of coequidistant bisectors bounded by Giraud discs. So two core face can, at worst, only meet in a two dimensional disc. We can then use the slice foliation of the bisectors to show that the intersection is exactly a triangle, a geodesic, a point or empty as in the combinatorial model.

We begin by proving several technical lemma that we will use to prove theorem 4.3.10.

Definition 4.3.14 *We denote the orthogonal projection onto the complex spine of* $\boldsymbol{A}$ *by* $\Pi_A : \boldsymbol{H}^2_{\mathbb{C}} \to \Sigma_A$*. Similarly* Π_B *and* Π_C*.*

Lemma 4.3.15 *The image of* $\boldsymbol{A}$ *under orthogonal projection onto its complex spine* Σ_A *is the geodesic segment* $[p_{1231}, p_{23}]$*. More succinctly,* $\Pi_A(\boldsymbol{A}) = [p_{1231}, p_{23}]$*. We will refer to this geodesic segment as the* ***relevant part of the real spine*** σ_A*.*

Corollary 4.3.16 *The image of* $\boldsymbol{B}$ *(respectively* $\boldsymbol{C}$*) under orthogonal projection onto its complex spine* Σ_B *(respectively* Σ_C*) is the geodesic segment* $[p_{1232}, p_{131212}]$ *(respectively* $[p_{2321232121}, p_{12321323}]$*).*

Lemma 4.3.17 *Let* γ *be (an arc of) a geodesic, then either* γ *is contained in* $\mathcal{B}_A$ *or* γ *intersects* $\boldsymbol{A}$ *in fewer points than* $\Pi_A(\gamma)$ *intersects* $[p_{1231}, p_{23}]$*.*

Proof: By proposition 1.2.27 in chapter 1, we know that the number of intersection points between a bisector and a geodesic is equal to the number of intersection points between the real spine of the bisector and the image of the geodesic under orthogonal projection onto the complex spine of the bisector. By 4.3.15 **A** is contained in the subset of the bisector $\mathcal{B}_A$ corresponding to the geodesic segment $[p_{1231}, p_{23}] \subset \sigma_A$. Assume that γ is not contained in a slice of $\mathcal{B}_A$, then γ intersects this subset of $\mathcal{B}_A$ in n points if the orthogonal projection of γ onto Σ_A intersects $[p_{1231}, p_{23}]$ in n points.

□

Corollary 4.3.18 *Let γ be (an arc of) a geodesic, then either γ is contained in $\mathcal{B}_B$ (respectively $\mathcal{B}_C$) or γ intersects $\boldsymbol{B}$ (respectively $\boldsymbol{C}$) in fewer points than $\Pi_B(\gamma)$ (respectively $\Pi_C(\gamma)$) intersects $[p_{1232}, p_{131212}]$ (respectively $[p_{2321232121}, p_{12321323}]$).*

This lemma and corollary provide us with an intuitive and relatively pain free way of proving that a geodesic cone face does not have any extra intersections with one of the core faces. We just need to ensure that the image of the geodesic cone face under orthogonal projection does not intersect badly with the relevant segment of the real spine.

Figures 4.6, 4.7 and 4.8 show the projection of the geodesic 1-skeleton of Δ onto the complex spines of $\mathcal{B}_A$, $\mathcal{B}_B$ and $\mathcal{B}_C$. The blue curves are the projections of the 1-skeleton of the core faces and the red curves are the projection of the geodesic segments between vertices of the core faces and the image of $*$, the black circle arc is the relevant part of the real spine. We have normalised so that the $\Pi_x(*)$ is at the origin. Recall that a $\mathbb{C}$-line in the ball is a copy of Poincaré disc model of $\mathbf{H}^2_{\mathbb{R}}$, so we can think of these pictures as curves and geodesics intersecting in the real hyperbolic plane.

The blue and red curves are the images of geodesics under orthogonal projection, so by 1.2.17, they are arcs of geometric circles (or geometric straight lines). It appears clear from the figures that the blue and red curves only intersect the black line in at most one end-point and careful analysis shows that these are exactly the points prescribed by the combinatorial model. These figures are not intended as a proof of 4.3.10 but they provide a good intuitive idea of what the proof involves- we show that the orthogonal projection of each cone face does not have any non-prescribed intersections with the real spine, so the cone face cannot have any non-prescribed intersection points with the core faces.

Our proof is conceptually similar to that of proposition 4.2 of [7]. The proof from that paper is somewhat lacking in detail; we go through the argument more thoroughly. In [7] it was possible to use the fact that the cone point was contained in the complex spines of the relevant bisectors to conclude that the orthogonal projection of any geodesic segment containing the cone point was itself a geodesic in the complex spine. This is not the case in our construction, however we can put precise bounds on how close the projected geodesic is to a geodesic and use these bounds to prove analogous results.

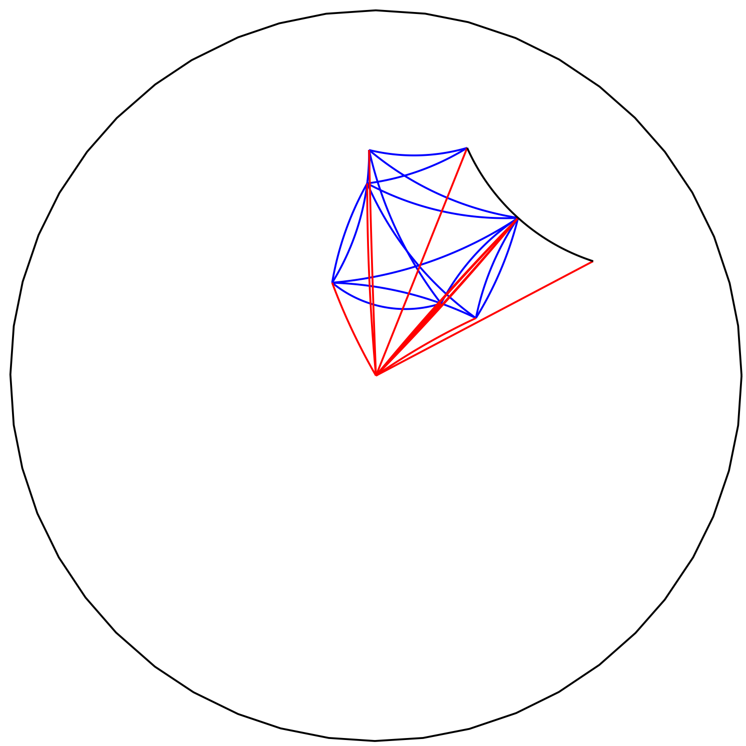

Figure 4.6: Projection of the 1-skeleton of Δ onto Σ_A

The first thing we need to do is ensure that the image of the triangular bases to the cone faces under orthogonal projection intersect the relevant part of the real spine correctly. By lemma 4.3.7 we know all the geodesic segments bounding a triangle base intersect with the core faces as expected, so we only need to check that the interior of the triangle has no extra intersections.

4.3.4 Giraud discs

Lemma 4.3.19 *Giraud discs admits three foliations by hypercycle segments.*

Proof: A Giraud disc, $\mathcal{G}$, is contained in the intersection of three bisectors $\mathcal{B}_1$, $\mathcal{B}_2$ and $\mathcal{B}_3$. Using the Mostow decomposition (theorem 1.2.25) each of these bisectors is foliated by $\mathbb{C}$-lines (the slices of $\mathcal{B}_i$). Let $\mathcal{C}_{1,p}$ be a slice of the bisector $\mathcal{B}_1$ corresponding to a point $p \in \sigma_1$, then we consider the intersection $\mathcal{C}_{1,p}$ with $\mathcal{G}$. By

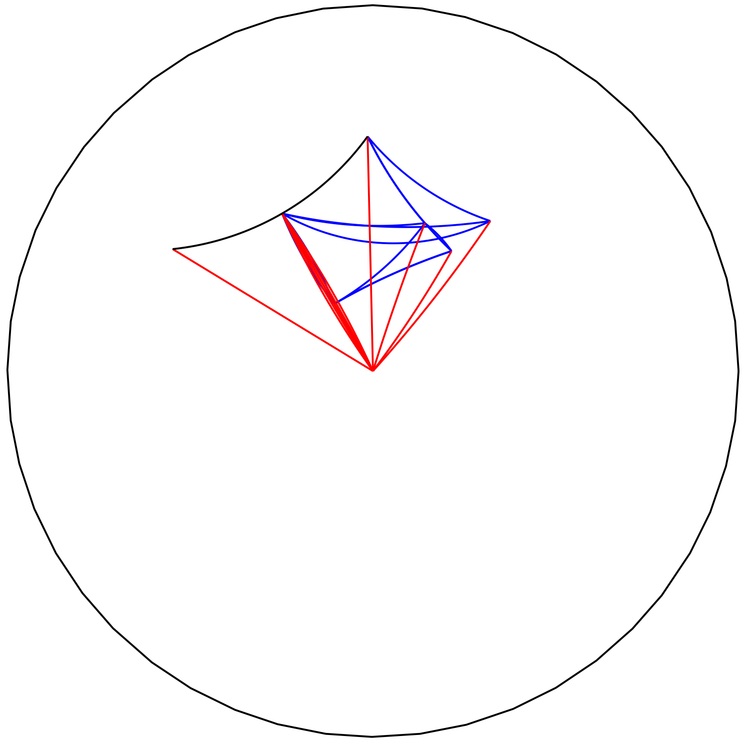

Figure 4.7: Projection of the 1-skeleton of Δ onto Σ_B

lemma 1.2.27 the intersection of a $\mathbb{C}$-line with a bisector is a hypercycle in $\mathcal{C}$. Since $\mathcal{G} = \mathcal{B}_1 \cap \mathcal{B}_2 \cap \mathcal{B}_3$, the intersection $\mathcal{C}_{1,p} \cap \mathcal{G}$ is a segment of a hypercycle in $\mathcal{C}_{1,p}$. Varying $p \in \sigma_1$ produces a family of hypercycle segments that foliates $\mathcal{G}$, we denote this foliation by F_1. Using the Mostow decomposition of the other two bisectors produces the other foliations, denoted F_2 and F_3. □

Definition 4.3.20 *Let $\mathcal{G}$ be a Giraud disc with hypercycle foliations F_1, F_2 and F_3 and D a subset of $\mathcal{G}$ homeomorphic to a 2-ball. We say that D is* ***Giraud convex*** *if, for all hypercycles $\delta \in F_i$, the intersection $\delta \cap \partial D$ consists of at most two points. Alternatively for all hypercycles $\delta \in F_i$, the intersection $\delta \cap D$ is either empty, a point or a connected subset of δ.*

Lemma 4.3.21 *A Giraud disc is always Giraud convex.*

Proof: This follows from Deraux analysis of Giraud discs in section 4 of [6]. □

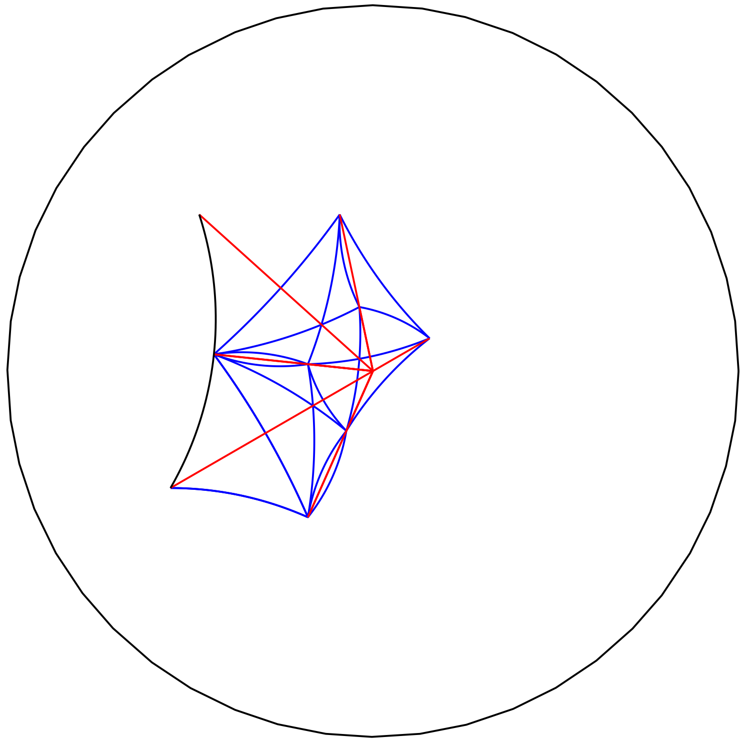

Figure 4.8: Projection of the 1-skeleton of Δ onto Σ_C

Definition 4.3.22 *Let $\mathcal{T}$ be a region of a Giraud disc whose boundary consists of three hypercycles from the foliations F_i. We call such a region a **Giraud triangle**. Note that necessarily the three bounding geodesics must come from three different foliations and $\mathcal{T}$ is Giraud convex.*

Lemma 4.3.23 *Except for the triangles* $(1232, 12321323, 2321232121)$ *and* $(12321231, 12321323, 2321232121)$ *all the triangles in the boundary of the core faces are Giraud triangles. The triangles* $(1232, 12321323, 2321232121)$ *and* $(12321231, 12321323, 2321232121)$ *are contained in $\mathbb{R}$-planes (meridians of $\mathcal{B}_C$).*

Proposition 4.3.24 *Let $\mathcal{D}$ be a Giraud convex region contained in $\mathcal{G}$, the Giraud disc that is the intersection of three equidistant bisectors $\mathcal{B}_1$, $\mathcal{B}_2$ and $\mathcal{B}_3$. Let $\mathcal{C}$ be a $\mathbb{C}$-line that is not the complex spine of $\mathcal{B}_1$, $\mathcal{B}_2$ or $\mathcal{B}_3$ and let $\Pi_{\mathcal{C}} : \boldsymbol{H}^2_{\mathbb{C}} \to \mathcal{C}$ be the orthogonal projection onto $\mathcal{C}$. Assume the projection $\Pi_{\mathcal{C}} : \partial\mathcal{D} \to \mathcal{C}$ is homeomorphic, then $\Pi_{\mathcal{C}} : \mathcal{D} \to \mathcal{C}$ is homeomorphic.*

If $\mathcal{C}$ is the complex spine of one of the bisectors, by theorem 1.2.25, the Giraud disc

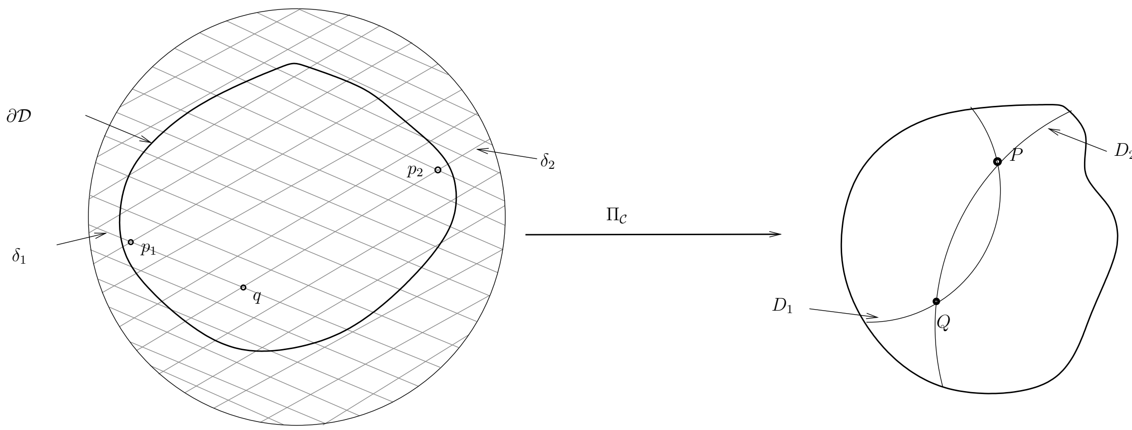

Figure 4.9: Orthogonal projection of a Giraud convex region

is mapped into the real spine.

PROOF: Let $\mathcal{D}$ be a Giraud convex region and $\mathcal{C}$ a $\mathbb{C}$-line such that $\Pi_{\mathcal{C}} : \partial\mathcal{D} \to \mathcal{C}$ is homeomorphic. Let F_1 and F_2 be two foliations of a Giraud disc $\mathcal{D}$. By proposition 1.2.17, in the ball model a hypercycle is mapped to the arc of a circle under orthogonal projection, so a hypercycle in F_1 or F_2 is sent to the arc of a circle when we orthogonally project the disc onto $\mathcal{C}$.

Assume that two points p_1 and p_2 in the interior of $\mathcal{D}$ are sent to the same point, $\mathbf{P}$, in $\Pi_{\mathcal{C}}(\mathcal{D})$. Then we can consider the third point, q, shown in figure 4.9. We define this point as follows, let δ_1 be the hypercycle in F_1 containing p_1 and δ_2 be the hypercycle in F_2 containing p_y (if necessary swapping the indices on F_1 and F_2). Let q be the point contained in the intersection of δ_1 and δ_2. This new point q is not equal to p_1 or p_2 since that would imply that p_1 and p_2 lie in the same hypercycle $\widehat{\delta}$ in either F_1 or F_2. Then when we orthogonally project $\widehat{\delta}$, the entire hypercycle including its end points $\widehat{\delta} \cap \partial\mathcal{D}$ will be sent to the same point (by lemma 1.2.16), so Π_C is not injective on the boundary.

We now consider the image of q under orthogonal projection. Since q and p_1 are contained in the same hypercycle, they cannot be mapped to the same point, similarly q and p_2. So the image of q under orthogonal projection, $\mathbf{Q}$, must be distinct from $\mathbf{P}$. Now we consider the image of the geodesics δ_1 and δ_2 under orthogonal projection, which we denote $\mathbf{D}_1$ and $\mathbf{D}_2$. By proposition 1.2.17, $\mathbf{D}_1$ and

$\mathbf{D}_2$ are arcs of circles. We know they intersect at $\mathbf{P}$ and $\mathbf{Q}$, so they cannot intersect anywhere else. The end points of either of the circle arcs lie on opposite sides of the other circle arc, so the circle arcs must intersect an odd number of times, but they cannot intersect more than twice, so they must intersect only once, therefore $\mathbf{P}$ and $\mathbf{Q}$ are the same point. This is a contradiction, therefore the images p_1 and p_2 under orthonormal projection onto $\mathcal{C}$ are distinct and $\Pi_{\mathcal{C}}$ is injective. □

PROOF:[of theorem 4.3.10] We are now ready to prove theorem 4.3.10. We reduce the problem to a lengthy, but elementary, argument about the intersection properties of circles in the complex plane, using lemmas 1.2.17 to 1.2.22 from chapter 1.

We shall only go through the details for $\mathbf{A}$ in detail, the other cases are essentially identical.

The core face $\mathbf{A}$ is contained in $\mathcal{B}_A$ with complex spine Σ_A, real spine σ_A and we denote the relevant part of the real spine $\widehat{\sigma_A}$. We normalise so that $\Sigma_A = \{(\zeta,0,1)^t | \zeta \in \mathbb{C}\}$, $\Pi_A(*) = (0,0,1)^t$ and the ideal endpoints of σ_A are $(e^{i\theta},0,1)^t$ and $(-e^{-i\theta},0,1)^t$.

Let p be a point in $\mathbf{H}^2_{\mathbb{C}}$, in this normalisation $\Pi_A([p,*]) = (\widehat{S_p},0,1)^t$, where $\widehat{S_p}$ is the shorter arc of circle between $\Pi_A(*)$ and $\Pi_A(p) = (X_p,0,1)^t$ (see lemma 1.2.17).

Note that there are two constants corresponding to this configuration of Σ_A, σ_a and $*$. These are θ, the argument of on of the end points of σ_A and $k = \sqrt{(l-1)/l}$ where

$$l = \frac{\langle *, \Pi_A(*)\rangle \langle \Pi_A(*), *\rangle}{\langle *, *\rangle \langle \Pi_A(*), \Pi_A(*)\rangle}.$$

Note that this k agrees with the k from lemma 1.2.17.

Explicitly in this normalisation, for Σ_A, σ_A and $*$, $\theta = 1.052705\ldots$, $k = 0.158918\ldots$ and $\sigma_A \subset (1/\sin(\theta))(\cos(\theta)e^{it} + i)$.

Let $\mathbf{F}$ be a cone face, then $\mathbf{F} = \mathrm{Cone}_*(\mathcal{T})$ for some triangle $\mathcal{T}$. By lemma 4.3.7, $\Pi_A(\partial\mathcal{T})$ lie on the same side of σ_A as $\Pi_A(*)$ in Σ_A. Then we use lemma 4.3.24 to conclude that $\Pi_A(\mathcal{T})$ does not intersect σ_A and lies entirely on the same side as $\Pi_A(*)$ (see remark 4.3.4 below).

Now we have to show that for any point $p \in \mathcal{T}$, the geodesic segment $[p,*]$ does not intersect $\mathbf{A}$ except where prescribed. We do this by showing that $\Pi_A([p,*])$ does not intersect $\widehat{\sigma_A}$.

By lemma 1.2.17, $\Pi_A([p,*])$ is the arc of a circle passing through the origin. We denote this circle S_p and the arc $\widehat{S_p}$. The arc $\widehat{S_p}$ is always the shorter of the two arc with end point $\Pi_A(*)$ and $\Pi_A(p)$.

We now use lemma 1.2.22, this describes a curve T_θ that divides Σ_A into two parts. If $\Pi_A(p)$ lies on the same side of T_θ as $\Pi_A(*)$, then $(\widehat{S_p},0,1)^t = \Pi_A([p,*])$ does not intersect σ_A (except possibly at $\Pi_A(p)$ if $p \in \mathcal{B}_A$). So $[p,*]$ does not intersect not intersect $\mathbf{A}$ (except at p if $p \in \mathbf{A}$).

This curve is shown in figure 4.10 along with the projection of the 1-skeleton of the core faces.

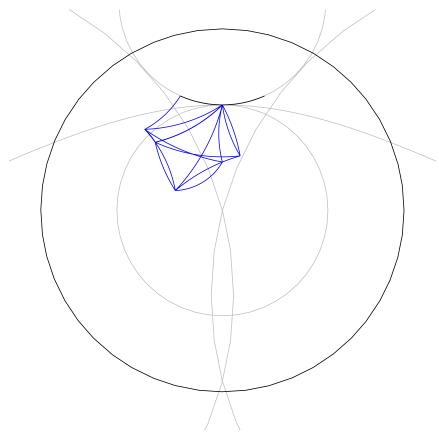

Figure 4.10: Projection of the 1-skeleton of Δ onto Σ_A with extra curves

By lemma 4.3.24 the 2-skeleton of the core faces (i.e. the union of the triangular bases) project onto the interior of the projection of the 1-skeleton. So for most $p \in \mathcal{T}$, the image $\Pi_A(p)$ lie on the good side of T_θ and therefore the geodesic segment $[p,*]$ cannot intersect $\mathbf{A}$.

There is a small region on the wrong side of T_θ see figure 4.11. Let p be a point such that $\Pi_A(p) = (X_p,0,1)^t$ lies in this bad region. Then $|X_p| > 0.581027\ldots = (1-\cos(\theta))/\sin(\theta)$, this is the radius of the grey circle in figure 4.10

Then by lemma 1.2.20 the radius for the circle arc through this point would have to be greater than $(1/2k)|X_p| \geq 1.828074\ldots$. In figures 4.10 and 4.11, the two larger circles passing through the origin are the circles with radius $1.828074\ldots$ that are

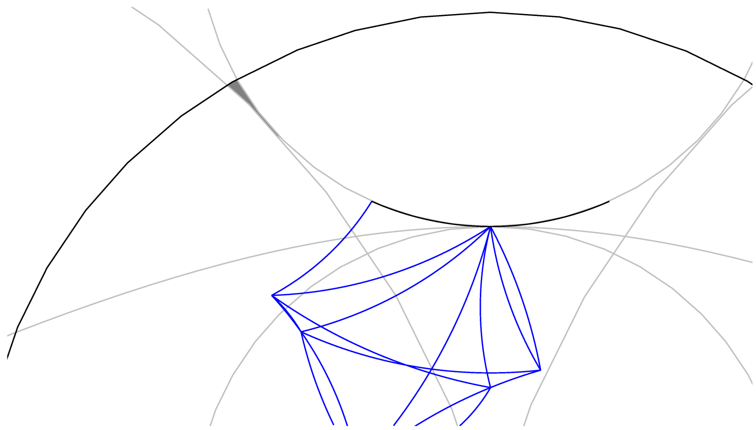

Figure 4.11: Close up of the projection of the 1-skeleton of Δ onto Σ_A

tangent to σ_A.

Then it follows from straightforward trigonometry that $\widehat{S_p}$ can only intersect σ_A if $\Pi_A(p)$ lies in the shaded region of figure 4.11. Since this does not happen, we conclude $\Pi_A([p,*])$ does not intersect σ_A except where prescribed by the combinatorial model so, by 4.3.17, $[p,*]$ does not intersect $\mathbf{A}$ except where prescribed by the combinatorial model.

An essentially identical argument shows that no cone faces intersect $\mathbf{B}$ except where specified, the projection of the 1-skeleton and the relevant curves are shown in figure. For $\mathbf{C}$ the situation is even simpler, since $*$ is contained in Σ_C, so $\Pi_C([p,*])$ is a geodesic segment and the projection of the geodesic cone over $\mathcal{T}$ is the geodesic cone over the projection of $\mathcal{T}$. Since $\Pi_C(\mathcal{T})$ lies on the same side of σ_A and $\Pi_C(*)$, it follows that $\mathrm{Cone}_*(\mathcal{T})$ does not intersect $\mathcal{C}$.

For Σ_B the values of θ and k are $0.922850\ldots$ and $0.225277\ldots$ respectively. The image of the projection are shown in figures 4.12 and 4.13.

For Σ_C the values of θ and k are $0.824783\ldots$ and 0 respectively. Since $k = 0$, the point $*$ is contained in Σ_C, any geodesic containing $*$ is projected onto a geodesic

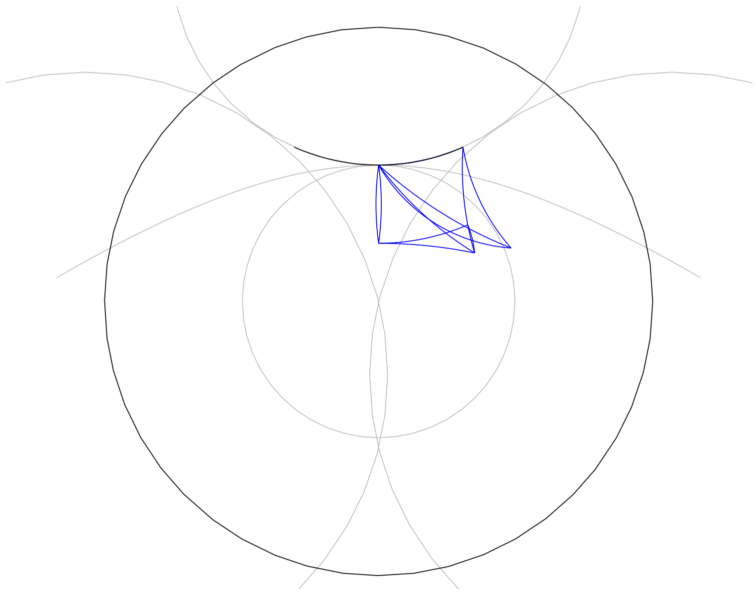

Figure 4.12: Projection of the 1-skeleton of Δ onto Σ_B with extra curves

through $\Pi_C(*)$. So we don't need to use the above argument. If $\Pi_C(p)$ lies on the same side of σ_C as $\Pi_C(*)$, then $\Pi_C([p,*])$ is a geodesic segment in Σ_C, so it cannot intersect σ_C (except possibly at the end point $\Pi_C(p)$), therefore $[p,*]$ cannot intersect $\mathcal{B}_C$ (except possibly at the end point p). $\square$

Corollary 4.3.25 *The interior of polyhedron Δ lies entirely on the good sides of $\mathcal{B}_A$, $\mathcal{B}_B$ and $\mathcal{B}_C$.*

Proof: This follows immediately from the proof of the previous theorem. In that proof we actually proved a stronger condition, namely that the polyhedron doesn't intersect the entire bisector $\mathcal{B}_A$ (respectively $\mathcal{B}_B$, $\mathcal{B}_C$) except where prescribed, rather than just the relevant part. Since the prescribed intersections are at most 3-dimensional it follows that the interior of Δ does not intersect $\mathcal{B}_A$ (respectively $\mathcal{B}_B$, $\mathcal{B}_C$) at all an consequently is entirely contained in the good side. $\square$

Remark: As it stands this proof is somewhat dependant on the accuracy of the pictures drawn by Maple. However it is possible to prove the result without relying

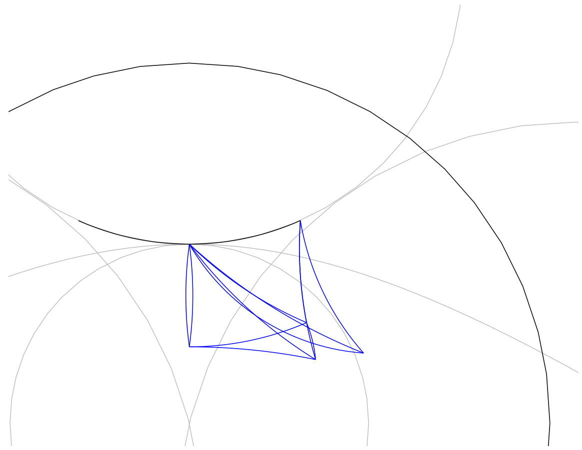

Figure 4.13: Close up of the projection of the 1-skeleton of Δ onto Σ_B

on Maple. This could in principle be done by explicitly calculating the images under orthogonal projection of each geodesic segment in the 1-skeleton of the core faces and checking their intersections with σ_A, T_θ and the three other circles. This amounts to checking that a large, but finite, number of esoteric inequalities hold.

Remark: In the proof of theorem 4.3.10, we assumed that the image of any of the triangle bases was projected into the interior of the image of the boundary, invoking proposition 4.3.24 to justify this assumption. There are two minor issues with this assumption, firstly the triangles $(1232, 12321323, 2321232121)$ and $(12321231, 12321323, 2321232121)$ are not Giraud triangles and we can't use proposition 4.3.24. However these triangles are contained in $\mathbb{R}$-plane and the projection of $\mathbb{R}$-planer triangles is much more straightforward than Giraud triangles. The orthogonal projection of a polygon in an $\mathbb{R}$-plane onto a $\mathbb{C}$-line is either a geodesic or a polygon or two polygons with a common vertex (Lemma 2.1 of [7]). So as before we only need to understand the behaviour of projection on the boundary of the polygon to extrapolate to the interior of the polygon.

The second issue is slightly more serious; what happens if $\Pi : \partial\mathcal{T} \to \mathcal{C}$ is not injective (so $\Pi : \mathcal{T} \to \mathcal{C}$ is not injective). In the cases we have just dealt with either this does occur or does so in way that is easily dealt with e.g on of the edges of a triangle is contained is slice of the bisector onto whose spine we are projecting, this mean the edge collapses to a single point but a similar argument to proof of proposition 4.3.24

can be used to show that projection is still homeomorphic on the interior.

4.4 Poincaré's polyhedron theorem

Having shown that our modified polyhedron Δ is homeomorphic to the combinatorial model we now use a form of Poincarés polyhedron Theorem (we follow the formulation given in [7]), to show that Δ is a fundamental domain for the group and extract a presentation for the group.

4.4.1 Statement of Poincaré's polyhedron theorem

Definition 4.4.1 *A Polyhedron is a cellular complex homeomorphic to a (compact) polytope, in particular we require that there is only one cell of highest dimension and that each codimension-two cell is contained in exactly two codimension-one cells. Then its realization as a cell complex in a space X is called a polyhedron.*

Definition 4.4.2 *A Poincaré polyhedron is a smooth polyhedron $D \subset X$, with the following conditions on the codimension-one faces T_i*

- *There is a set Λ of homeomorphisms of X of the form $A_{ij} : T_i \to T_j$, which pairs the codimension-one faces, moreover these maps preserve the cell structure on D. We call these maps* side-pairing transformations*. We assume that if $A_{ij} \in \Lambda$ then $A_{ij}^{-1} = A_{ji} \in \Lambda$.*
- *For every $A_{ij} \in \Lambda$ such that $A_{ij}(T_i) = T_j$, then $A_{ij}\left(\overline{D}\right) \cap \overline{D} = T_j$.*
- *If $T_i = T_j$ (i.e. if A_{ij} sends T_i to itself), then we impose that $A_{ii} : T_i \to T_i$ is of order two and we call it a* reflection*.*

Let the pair (F_1, T_1) be a codimension-two face and a codimension-one face such that $F_1 \subset T_1$. Then there exists exactly one other codimension-one face, T_1', that also contains F_1 and a side paring map $g_1 \in \Lambda$ associated to T_1'. Let $F_2 = g_1(F_1)$ and $T_2 = g_1(T_1')$. Again there exists exactly one other codimension-one face, T_2' containing F_2 and we can continue to recursively define such g_i and F_i so that $g_{i-1} \circ \cdots \circ g_1(F_1) = F_i$

Definition 4.4.3 ***Cyclic***

Given (F_1, T_1) *as above* ***Cyclic*** *states that there exists an* $r \geq 1$ *such that* $g_r \circ \cdots \circ g_1(T_1) = T_1$ *and* $g_r \circ \cdots \circ g_1$ *restricted to* F_1 *is the identity. Also letting* $g = g_r \circ \cdots \circ g_1$ *then there exists a positive integer* m *such that*

$$\begin{gathered} g_1^{-1}(D) \cup (g_2 \circ g_1)^{-1}(D) \cup \cdots \cup g^{-1}(D) \cup \\ (g_1 \circ g)^{-1}(D) \cup (g_2 \circ g_1 \circ g)^{-1}(D) \cup \cdots \cup (g^2)^{-1}(D) \cup \\ \vdots \\ \left(g_1 \circ g^{m-1}\right)^{-1}(D) \cup \left(g_2 \circ g_1 \circ g^{m-1}\right)^{-1}(D) \cup \cdots \cup (g^m)^{-1}(D) \end{gathered}$$

is a cover of a closed neighbourhood of the interior of F_1.

The relation $g^m = (g_r \circ \cdots \circ g_1)^m = Id$ *is called a* cycle relation.

Theorem 4.4.4 (Poincaré's polyhedron theorem) *Let* $D \subset \boldsymbol{H}^2_{\mathbb{C}}$ *be a compact Poincaré polyhedron with side pairing transformations* $\Lambda \subset \mathrm{Isom}\left(\boldsymbol{H}^2_{\mathbb{C}}\right)$ *satisfying* ***Cyclic****. Let* Γ *be the group generated by* Λ *then,*

- Γ *is a discrete subgroup of* $\boldsymbol{H}^2_{\mathbb{C}}$*,*
- D *is a fundamental domain for* Γ*, and*
- Γ *has the presentation:* $\Gamma = \langle \Lambda \mid \text{cycle relations},\ \text{reflection relations} \rangle$.

4.4.2 List of 3-faces and side pairings

To reduce the number of conditions that need to be checked we now glue together any 3-faces that have the same side pairing transformations

Core faces

These remain unchanged, i.e. **A**, **B** and **C** as in figure 4.1.

Prism faces

We combine the prism faces as follows,

$$\begin{aligned}
\mathbf{D}_1 =& \mathrm{Cone}_*(13, 12, 23) \cup \mathrm{Cone}_*(13, 23, 1323),\\
\mathbf{D}_2 =& \mathrm{Cone}_*(1231, 13, 12) \cup \mathrm{Cone}_*(1231, 12, 131212),\\
\mathbf{E}_1 =& \mathrm{Cone}_*(12, 23, 1232) \cup \mathrm{Cone}_*(12, 1232, 232121) \cup \mathrm{Cone}_*(12, 131212, 232121)\cup\\
& \mathrm{Cone}_*(131212, 232121, 2321232121),\\
\mathbf{E}_2 =& \mathrm{Cone}_*(1231, 13, 1323) \cup \mathrm{Cone}_*(1231, 1323, 1232) \cup \mathrm{Cone}_*(1231, 1232, 12321231)\cup\\
& \mathrm{Cone}_*(12321231, 1232, 12321323),\\
\mathbf{F}_1 =& \mathrm{Cone}_*(23, 1323, 1232),\\
\mathbf{F}_2 =& \mathrm{Cone}_*(1231, 12321231, 131212),\\
\mathbf{G}_1 =& \mathrm{Cone}_*(1232, 232121, 2321232121) \cup \mathrm{Cone}_*(1232, 12321323, 2321232121),\\
\mathbf{G}_2 =& \mathrm{Cone}_*(12321231, 131212, 2321232121) \cup \mathrm{Cone}_*(12321231, 12321323, 2321232121).
\end{aligned}$$

The side pairings of the Polyhedron Δ are

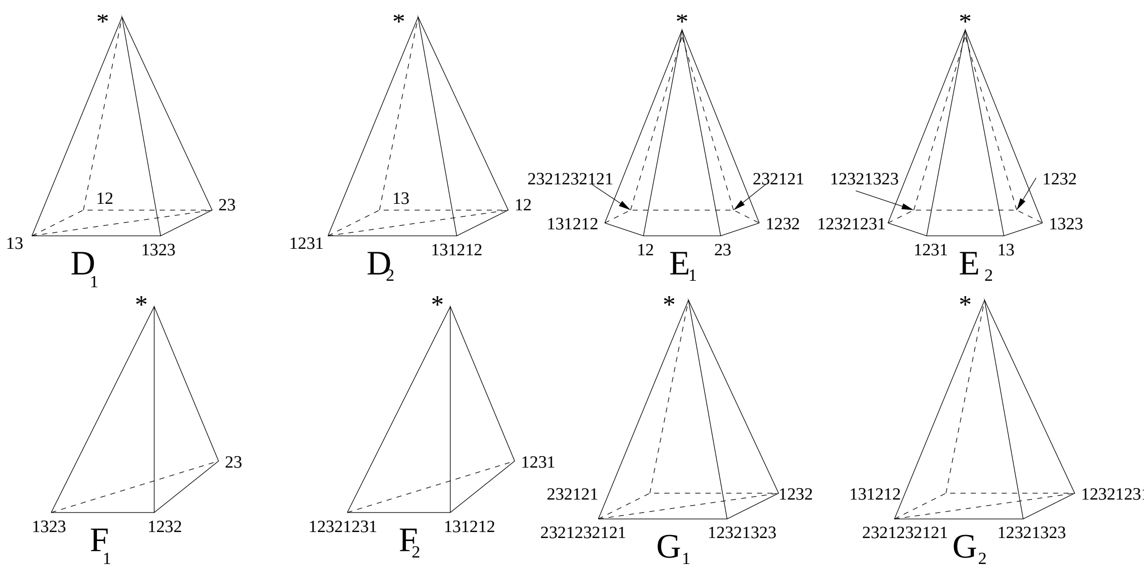

Figure 4.14: New 3-faces of Δ

$$\mathbf{A} \xrightarrow{1} \mathbf{A}, \quad \mathbf{B} \xrightarrow{121} \mathbf{B}, \quad \mathbf{C} \xrightarrow{123212321} \mathbf{C},$$
$$\mathbf{D}_1 \xrightarrow{P^{-1}} \mathbf{D}_2, \quad \mathbf{E}_1 \xrightarrow{P^{-2}} \mathbf{E}_2, \quad \mathbf{F}_1 \xrightarrow{P^{-3}} \mathbf{F}_2,$$
$$\mathbf{G}_1 \xrightarrow{P^{-5}} \mathbf{G}_2.$$

4.4.3 Cycles of 2-faces

As noted above, for reasons of simplicity, we consider the index three supergroup $\langle I_1, J\rangle$ rather than $(4,4,4;5)$. Consequently the reflection relations are $(1)^2 = (121)^2 = (123212321)^2 = Id$, conjugating by J we can recover $(2)^2 = (3)^2 = Id$. This allows to remove double letters in the cycle words to get the cycle relations shown in the table below

Face	Cycle length	Cycle relation
$(13, 12, 23)$	2 (6)	$J^3 = Id$
$(12, 23, 1232)$	5	Id
$(23, 1323, 1232)$	6	$23232323 = Id$
$(1232, 232121, 2321232121)$	5	$21212121 = Id$
$(12321231, 2321232121, 12321323)$	4	$132(1323)^3 12323212321 = Id$
$(23, 1323, *)$	3	Id
$(13, 1323, *)$	3	Id
$(12, 13, *)$	3	Id
$(1232, 232121, *)$	3	Id
$(232121, 2321232121, *)$	4	Id
$(2321232121, 12321323, *)$	1 (6)	$(J1)^{30} = Id$

Because of the cyclic symmetry J some of the above cycle relations are redundant, for example conjugating I_{23}^4 by J gives I_{31}^4, and I_{12}^4. We can obtain I_{1323}^5 from the relation in the fifth row and conjugation by J and I_i give the other relations of the form I_{ijkj}. Finally from a straightforward word manipulation we see that $(JI_1)^{30} = (I_{231})^{10}$, Then taking inverses or conjugates of these words we obtain all the relations shown in presentation (4.8).

4.5 Tessellations

We first ensure that the side pairings map the interior of Δ off itself.

Theorem 4.5.1 *Let* $g \in \{I_1,\ I_{121},\ I_{123212321},\ P^{-5},\ P^{-3},\ P^{-2},\ P^{-1}\}$. *Then* $g\left(\Delta^0\right) \cap \Delta^0 = \emptyset$.

We use Δ^0 to denote the interior of Δ.

PROOF: If $g = I_1$, (respectively I_{121}, $I_{123212321}$), then we can use the proof of 4.3.10. We proved that Δ does not intersect $\mathcal{B}_A$ (respectively $\mathcal{B}_B$, $\mathcal{B}_C$) except in $\mathbf{A}$ itself (respectively $\mathbf{B}$, $\mathbf{C}$), in particular Δ^0 lies entirely in the good half of $\mathbf{H}^2_{\mathbb{C}}$ with respect to $\mathcal{B}_A$ (respectively $\mathcal{B}_B$, $\mathcal{B}_C$). By construction I_1 (respectively I_{121}, $I_{123212321}$) fixes $\mathcal{B}_A$ (respectively $\mathcal{B}_B$, $\mathcal{B}_C$) and swaps the two half-spaces separated by the bisector. So $I_1(\Delta) \cap \Delta \subset \mathcal{B}_A$ therefore $I_1(\Delta^0) \cap \Delta^0 = \emptyset$. Similarly for I_{121} and $I_{123212321}$.

For $g = P^n$ the interior of $P^n(\Delta^0)$ consists of the geodesic cone over the interior of the core faces. The result follows from theorem 4.5.2 below. Assume $\Delta^0 \cap P^n(\Delta^0) \neq \emptyset$, then there exists a point $p \in P^n(\mathbf{A} \cap \mathbf{B})$ and a point $q \in (\mathbf{A} \cap \mathbf{B})$ such that $[p, *] \subset [q, *]$ or $[q, *] \subset [p, *]$. The former implies that the geodesic segment $[p, *]$ belonging to $P^n(\Delta^0)$ crosses $\mathcal{B}_A$, $\mathcal{B}_B$ or $\mathcal{B}_C$ and hence $P^n(\Delta^0)$ is not entirely contained in the at least one of the good halves of $\mathbf{H}^2_{\mathbb{C}}$ with respect to $\mathcal{B}_A$, $\mathcal{B}_B$ or $\mathcal{B}_C$. In the latter case the geodesic segment $[q, *]$ belonging to Δ^0 crosses $P^n(\mathcal{B}_A)$, $P^n(\mathcal{B}_B)$ or $P^n(\mathcal{B}_C)$. Applying P^{-n}, we get that $P^{-n}(\Delta^0)$ is not entirely contained in the at least one of the good halves of $\mathbf{H}^2_{\mathbb{C}}$ with respect to $\mathcal{B}_A$, $\mathcal{B}_B$ or $\mathcal{B}_C$. □

Theorem 4.5.2 *For $n \in \{0, \pm 1, \pm 2, \pm 3, \pm 5\}$ $P^n(\Delta^0)$ lies entirely on the good half of $\boldsymbol{H}^2_{\mathbb{C}}$ with respect to $\mathcal{B}_A$, $\mathcal{B}_B$ and $\mathcal{B}_C$.*

PROOF: See appendix A. □

In fact, this corollary is true for all $n \in \{0, \ldots, 29\}$, but we only need these nine values to confirm the tessellation conditions.

Our approach to checking the tessellation conditions follows that given in [7], first we show that all the polyhedra in a cycle have disjoint interiors, this is done by finding a separating bisector.

Once we have shown that all the polyhedra surrounding a 2-face have disjoint interiors we need to ensure that polyhedra cover a neighbourhood of the 2-face. For a general fundamental domain this can be very difficult to prove, however for Δ all the 2-faces are Giraud triangles, triangles contained in $\mathbb{R}$-planes or sides of cone faces, so we can use arguments similar to those used by Parker for fundamental domains of the Livné lattices [22].

Recall from proposition 1.2.33 in chapter 1, that a Giraud disc is contained in three bisectors and can be thought of as the locus of points equidistant from three distinct non-$\mathbb{C}$-linear points p_0, p_1 and p_2. We can use these points to divide $\mathbf{H}^2_{\mathbb{C}}$ into three regions

$$\{x \in \mathbf{H}^2_{\mathbb{C}} \ : \ d(x,p_0) \leq d(x,p_1) \text{ and } d(x,p_0) \leq d(x,p_2)\},$$
$$\{x \in \mathbf{H}^2_{\mathbb{C}} \ : \ d(x,p_1) \leq d(x,p_0) \text{ and } d(x,p_1) \leq d(x,p_2)\},$$
$$\{x \in \mathbf{H}^2_{\mathbb{C}} \ : \ d(x,p_2) \leq d(x,p_1) \text{ and } d(x,p_2) \leq d(x,p_0)\}.$$

Clearly every point in $\mathbf{H}^2_{\mathbb{C}}$ lies in at least of these regions. The intersection of two of these regions is precisely one of the three bisectors and the intersection of all three regions is the Giraud disc. We use this fact to confirm the tessellation satisfies **cyclic**.

Tessellation about $(13, 12, 23)$

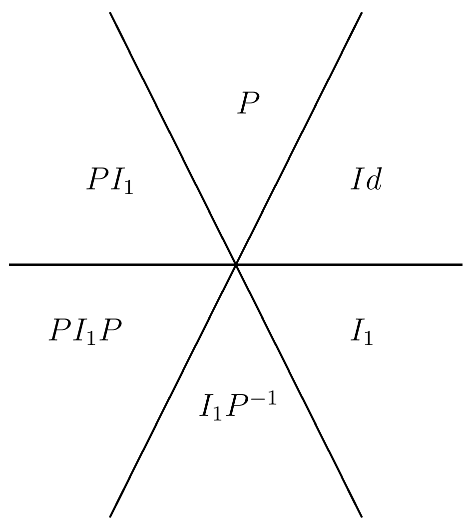

Figure 4.15: Schematic of tessellation about $(13, 12, 23)$

By 4.5.1, we know that $P(\Delta^0)$ and $I_1(\Delta^0)$ are disjoint with Δ^0. We must now compare all the 'diagonal' pairs. We can use the triple symmetry of J to reduce the number things of pairs the need to be check to $\Delta^0 \cap PI_1(\Delta^0)$,$\Delta^0 \cap PI_1P(\Delta^0)$ and $\Delta^0 \cap I_1P^{-1}(\Delta^0)$.

For $I_1 P^{-1}(\Delta^0$, recall that the bisector $\mathcal{B}_A$ separates $\mathbf{H}^2_{\mathbb{C}}$ into two half spaces one of which contains $*$, which we called the good side of $\mathcal{B}_A$. The interior of $P^{-1}(\Delta)$ is contained in this half space (by theorem 4.5.2). Since I_1 fixes $\mathcal{B}_A$ and swaps these half spaces, the interior $I_1 P^{-1}(\Delta)$ is contained in the other half space. In other words the bisector $\mathcal{B}_A$ separates Δ^0 and $I_1 P^{-1}(\Delta)$.

Since Δ^0 and $I_1 P^{-1}(\Delta^0)$ are disjoint, their images under the isometry $J = PI_1$ also have disjoint interiors, i.e. Δ^0 and $PI_1(\Delta^0)$ are disjoint.

This only leaves PI_1P. We know that $P(\Delta^0)$ and $P^{-1}(\Delta^0)$ both lie on the good side of $\mathcal{B}_A$, so $P^{-1}(\Delta^0)$ and $I_1 P(\Delta^0)$ have disjoint interiors. Then applying the map P we see that $(\Delta^0) \cap PI_1P(\Delta^0) = \emptyset$.

Applying J or J^{-1} to these relations shows that all the copies of Δ surrounding the 2-face $(13, 12, 23)$ have disjoint interiors.

We now have to show these six polyhedra cover a neighbourhood of the Giraud triangle $(13, 12, 23)$. We know by construction the Giraud disc containing $(13, 12, 23)$ is the intersection of the bisectors $\mathcal{B}_A$ and $P(\mathcal{B}_A)$ so we can define the point p_0 to be $\Sigma_A \cap P(\Sigma_A)$. From this we can define the other points $p_1 = J(p_0)$ and $p_2 = J^{-1}(p_0)$.

We can use the inequalities described above to divide $\mathbf{H}^2_{\mathbb{C}}$ into three regions

$$
\begin{aligned}
\mathcal{H}_0 &= \{x \in \mathbf{H}^2_{\mathbb{C}} \ : \ d(x, p_0) \le d(x, p_1) \text{ and } d(x, p_0) \le d(x, p_2)\},\\
\mathcal{H}_1 &= \{x \in \mathbf{H}^2_{\mathbb{C}} \ : \ d(x, p_1) \le d(x, p_0) \text{ and } d(x, p_1) \le d(x, p_2)\},\\
\mathcal{H}_2 &= \{x \in \mathbf{H}^2_{\mathbb{C}} \ : \ d(x, p_2) \le d(x, p_1) \text{ and } d(x, p_2) \le d(x, p_0)\}.
\end{aligned}
$$

We combine the six polyhedra into the following three larger polyhedra $\Delta \cup P(\Delta)$, $I_1(\Delta) \cup I_1 P^{-1}(\Delta)$ and $PI_1(\Delta) \cup PI_1P(\Delta)$. By construction $\mathcal{H}_0 \cap \mathcal{H}_2 = \mathcal{B}_A$ and $\mathcal{H}_0 \cap \mathcal{H}_1 = P(\mathcal{B}_A)$. The other intersection is the third bisector and the triple intersection is the Giraud disc containing the triangle $(13, 12, 23)$.

If a point z is sufficiently close to $(13, 12, 23)$ and also lies in the region $\mathcal{H}_0$ then by the above argument about about separating bisectors it must be contained in $\Delta \cup P(\Delta)$. Similarly if z is sufficiently close to $(13, 12, 23)$ and also lies in the region $\mathcal{H}_1$ (respectively $\mathcal{H}_2$) it is contained in $I_1(\Delta) \cup I_1 P^{-1}(\Delta)$ (respectively $PI_1(\Delta) \cup PI_1P(\Delta)$)

From this we conclude the tessellation about $(13, 12, 23)$ satisfies the condition

cyclic.

Remark: When the 2-face about which we are tessellating is a triangle contained in a Giraud disc we can always use the structure of the three bisectors to ensure that neighbourhood of the 2-face is covered. The argument is essentially identical to the $(13, 12, 23)$ case outlined above.

Tessellation about $(12, 23, 1232)$

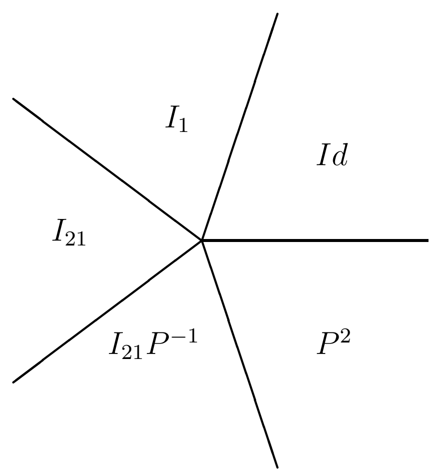

Figure 4.16: Schematic of tessellation about $(12, 23, 1232)$

By 4.5.1, we know the interiors of adjacent polyhedra are disjoint since they differ by a side pairing. We have to check the five diagonal pairs $\Delta^0 \cap I_{21}(\Delta^0)$, $\Delta^0 \cap I_{21}P^{-1}(\Delta^0)$, $I_1(\Delta^0) \cap P^2(\Delta^0)$, $I_1(\Delta^0) \cap I_{21}P^{-1}(\Delta^0)$ and $I_{21}(\Delta^0) \cap P^2(\Delta^0)$.

Claim: The interiors of $I_1(\Delta)$ and $P^2(\Delta)$ are disjoint.

We know that $P^2(\Delta)$ is contained in the good side of $\mathcal{B}_A$ and $I_1(\Delta)$ is contained in the bad side, so they can only intersect in $\mathcal{B}_A$ itself hence $I_1(\Delta^0) \cap P^2(\Delta^0) = \emptyset$.

Claim: The interiors of Δ and $I_{21}(\Delta)$ are disjoint.

Note that the 2-face $(12, 23, 1232)$ is a triangle contained in a Giraud disc, as such it is contained in the intersection of exactly 3 bisectors. We know two of these bisectors namely $\mathcal{B}_A$ and $P^2(\mathcal{B}_A)$. The bisector $P^2(\mathcal{B}_A)$ separates Δ^0 and $I_{21}(\Delta^0)$. To see this we first observe that $P^{-2}(\Delta^0)$ and $I_1P(\Delta^0)$ lie on opposite sides of the bisector $\mathcal{B}_A$. We can rewrite $I_1P = I_1JI_1 = JI_3I_1 = JI_3I_2I_2I_1 \;\; P^{-2}I_2I_1$. S the bisector $\mathcal{B}_A$ separates $P^{-2}(\Delta^0)$ and $P^{-2}I_2I_1(\Delta^0)$. Applying the map P^2 proves the claim.

Claim: The interiors of Δ and $I_{21}P^{-1}(\Delta)$ are disjoint.

Again we show that $P^2(\mathcal{B}_A)$ separates Δ^0 and $I_{21}P^{-1}(\Delta^0)$, this is more straightforward than the previous case. We know that $\mathcal{B}_A$ separates $I_1(\Delta^0)$ and $P^{-2}(\Delta^0)$, so $P^2(\mathcal{B}_A)$ separates $P^2I_1(\Delta^0)$ and (Δ^0). The result follows from the fact $P^2I_1 = I_{21}P^{-1}$.

Claim: The interiors of $I_{21}(\Delta)$ and $P^2(\Delta)$ are disjoint.

We know that $P^2(\mathcal{B}_A)$ separates Δ^0 and $I_{21}(\Delta^0)$, in particular, $I_{21}(\Delta^0)$ lies entirely on one side of $P^2(\mathcal{B}_A)$. We also know that Δ^0 lies entirely on one side of $\mathcal{B}_A$ and hence $P^2(\Delta^0)$ lies entirely on one side of $P^2(\mathcal{B}_A)$. Then checking the position of $P^2(*)$ and $I_{21}(*)$ we see that $P^2(\mathcal{B}_A)$ separates $P^2(\Delta^0)$ and $I_{21}(\Delta^0)$ as required

Claim: The interiors of $I_1(\Delta)$ and $I_{21}P^{-1}(\Delta)$ are disjoint.

The image of this pair under I_{12} is $I_{121}(\Delta^0) \cap P^{-1}(\Delta^0)$. The polyhedra Δ^0 and $P^{-1}(\Delta^0)$ are contained entirely on the good side of the bisector of $\mathcal{B}_B$ and I_{121} fixes $\mathcal{B}_B$ while swapping the two half spaces. Therefore $I_{121}(\Delta^0)$ and $P^{-1}(\Delta^0)$ lie on opposite sides of the bisector $\mathcal{B}_B$ and $I_1(\mathcal{B}_B)$ separates the interiors of $I_1(\Delta)$ and $I_{21}P^{-1}(\Delta)$.

Tessellation about $(23, 1323, 1232)$

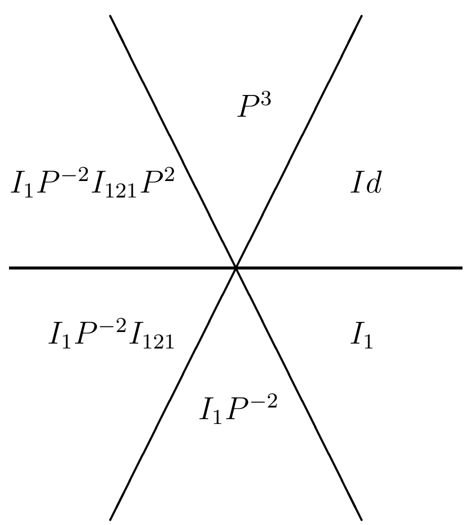

Figure 4.17: Schematic of tessellation about $(23, 1323, 1232)$

For this cycle there are 9 pairs to check (in general an n-cycle will have $\triangle(n) - n$

pairs to check, where $\triangle(n)$ is the nth triangular number).

The nine pairs are

1. $\Delta^0 \cap I_1 P^{-2} I_{121} P^2(\Delta^0)$
2. $\Delta^0 \cap I_1 P^{-2} I_{121}(\Delta^0)$
3. $\Delta^0 \cap I_1 P^{-2}(\Delta^0)$
4. $P^3(\Delta^0) \cap I_1 P^{-2} I_{121}(\Delta^0)$
5. $P^3(\Delta^0) \cap I_1 P^{-2}(\Delta^0)$
6. $P^3(\Delta^0) \cap I_1(\Delta^0)$
7. $I_1(\Delta^0) \cap I_1 P^{-2} I_{121} P^2(\Delta^0)$
8. $I_1(\Delta^0) \cap I_1 P^{-2} I_{121}(\Delta^0)$
9. $I_1 P^{-2}(\Delta^0) \cap I_1 P^{-2} I_{121} P^2(\Delta^0)$

For most of these pairs we can immediately find a separating bisector.

Pair 3: $\Delta^0 \cap I_1 P^{-2}(\Delta^0)$. This follows from the fact $\mathcal{B}_A$ separates $I_1(\Delta^0)$ and $P^{-2}(\Delta^0)$ and I_1 fixes $\mathcal{B}_A$ and swaps the two half spaces. Therefore $\mathcal{B}_A$ separates Δ and $I_1 P^{-2}(\Delta)$.

Pair 5: $P^3(\Delta^0) \cap I_1 P^{-2}(\Delta^0)$. The polyhedra $P^3(\Delta^0)$ and $P^{-2}(\Delta^0)$ both lie on the same side of $\mathcal{B}_A$, so $\mathcal{B}_A$ separates $P^3(\Delta^0)$ and $I_1 P^{-2}(\Delta^0)$.

Pair 6: $P^3(\Delta^0) \cap I_1(\Delta^0)$. Again the separating bisector is obviously $\mathcal{B}_A$.

Pair 7: $I_1(\Delta^0) \cap I_1 P^{-2} I_{121} P^2(\Delta^0)$. After applying the map $P_2 I_1$ to this pair we get $P^2(\Delta^0) \cap I_{121} P^2(\Delta^0)$. We know $P^2(\Delta^0)$ lies on the good side of $\mathcal{B}_B$ and I_{121} fixes $\mathcal{B}_B$ and swap the two corresponding half spaces. So $P^2(\Delta^0)$ and $I_{121} P^2(\Delta^0)$ are separated by $\mathcal{B}_B$. Then it follows $I_1 P^{-2} \mathcal{B}_B$ separates $I_1(\Delta^0)$ and $I_1 P^{-2} I_{121} P^2(\Delta^0)$.

Pair 8: $I_1(\Delta^0) \cap I_1 P^{-2} I_{121}(\Delta^0)$. Again we apply the map $P_2 I_1$ to this pair to get $P^2(\Delta^0) \cap I_{121}(\Delta^0)$. The separating bisector for this pair is $\mathcal{B}_B$. Therefore the bisector $I_1 P^{-2} \mathcal{B}_B$ separates $I_1(\Delta^0)$ and $I_1 P^{-2} I_{121}(\Delta^0)$.

Pair 9: $I_1 P^{-2}(\Delta^0) \cap I_1 P^{-2} I_{121} P^2(\Delta^0)$. Apply the map $P_2 I_1$ to get $I_{121} P^2(\Delta^0) \cap (\Delta^0)$. As before the bisector $I_1 P^{-2} \mathcal{B}_B$ separates $I_1 P^{-2}(\Delta^0)$ and $I_1 P^{-2} I_{121} P^2(\Delta^0)$.

This just leaves pairs 1, 2 and 4. Recall that the 2-face $(23, 1323, 1232)$ is contained in the Giraud disc $\mathcal{B}_A \cap P^3(\mathcal{B}_B)$.

Pair 1: $\Delta^0 \cap I_1 P^{-2} I_{121} P^2(\Delta^0)$. These polyhedra are separated by the bisector $P^3(\mathcal{B}_B)$.

Pair 2: $\Delta^0 \cap I_1 P^{-2} I_{121}(\Delta^0)$. These polyhedra are separated by the bisector $P^3(\mathcal{B}_B)$.

Pair 4: $P^3\Delta^0 \cap I_1 P^{-2} I_{121}(\Delta^0)$. These polyhedra are separated by the bisector $P^3(\mathcal{B}_B)$. This follows immediately from pair 2; Δ^0 and $P^3(\Delta^0)$ lie on the same side of $P^3(\mathcal{B}_B)$, so if $P^3(\mathcal{B}_B)$ separates Δ^0 and $I_1 P^{-2} I_{121}(\Delta^0)$ it must also separate Δ^0 and $I_1 P^{-2} I_{121}(\Delta^0)$.

Tessellation about $(1232, 232121, 2321232121)$

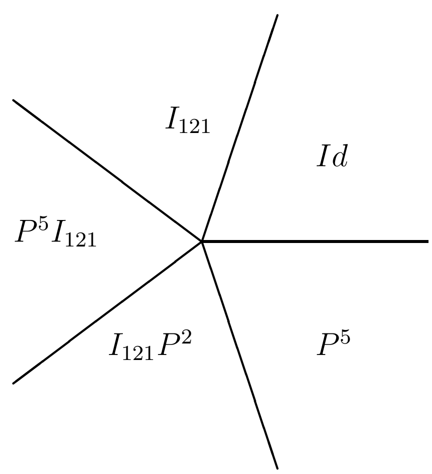

Figure 4.18: Schematic of tessellation about $(1232, 232121, 2321232121)$

This cycle has length five, so there are five diagonal pairs to check $\Delta^0 \cap P^5 I_{121}(\Delta^0)$, $\Delta^0 \cap I_{121}P^2(\Delta^0)$, $I_{121}(\Delta^0) \cap P^5(\Delta^0)$, $I_{121}(\Delta^0) \cap I_{121}P^2(\Delta^0)$ and $P^5(\Delta) \cap I_{121}(\Delta^0)$. Four of these intersections are empty follows immediately from the fact that $\mathcal{B}_B$ separates $I_{121}(\Delta^0)$ from $P^n(\Delta^0)$. The only remaining case is $I_{121}(\Delta^0) \cap I_{121}P^2(\Delta^0)$, applying I_{121} to both sides produces $\Delta^0 \cap P^2(\Delta^0)$ and we know this intersection is empty by 4.5.2.

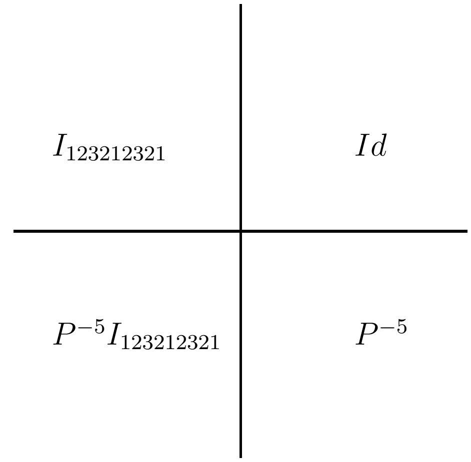

Figure 4.19: Schematic of tessellation about $(1232, 232121, 2321232121)$

Tessellation about $(12321231, 2321232121, 12321323)$

First observe that P^5 and $I_{123212321}$ commute. By 4.5.1, we know the interiors of $I_{123212321}(\Delta)$ and $P^{-5}(\Delta)$ are disjoint with the interior of Δ. We have to check the diagonal image $P^{-5}I_{123212321}(\Delta)$. We know from 4.5.1 that $P^{-5}(\Delta^0)$ lies entirely on the good side of $\mathcal{B}_C$ and $I_{123212321}$ fixes $\mathcal{B}_C$, whole swapping the two half spaces. So $\mathcal{B}_C$ separates $I_{123212321}(\Delta^0) \cup P^{-5}I_{123212321}(\Delta^0)$ from $P^{-5}(\Delta^0) \cup \Delta^0$.

This 2-face is contained in a $\mathbb{R}$-plane rather than a Giraud disc, the $\mathbb{R}$-plane is a meridian of $\mathcal{B}_C$. This makes the situation somewhat simpler than the Giraud disc cases. We can use the bisector structure of $\mathcal{B}_C$ to check the polyhedra cover a neighbourhood of $(12321231, 2321232121, 12321323)$. The bisector $\mathcal{B}_C$ separates $\mathbf{H}^2_{\mathbb{C}}$ into two half spaces, any point in a neighbourhood of $(12321231, 2321232121, 12321323)$ is contained in exactly one these half spaces or the triangle itself. If the point is sufficiently close to the triangle and in the good half of $\mathcal{B}_C$ then it is contained in $\Delta \cup P^{-5}(\Delta)$. If it is in the other half space then it is contained in $I_{123212321}(\Delta) \cup P^{-5}I_{123212321}(\Delta)$.

Tessellations about $(23, 1323, *)$**,** $(13, 1323, *)$**,** $(12, 13, *)$ **and** $(1232, 232121, *)$

These cycle all have total length 3 so, by 4.5.1, there is nothing to check.

We now check that the three polyhedra cover a neighbourhood of the 2-faces. Note that the 2-faces are not contained in a Giraud disc or a totally geodesic subspaces. We will only go through the details for $(23, 1323, *)$ since the other cases are essentially identical.

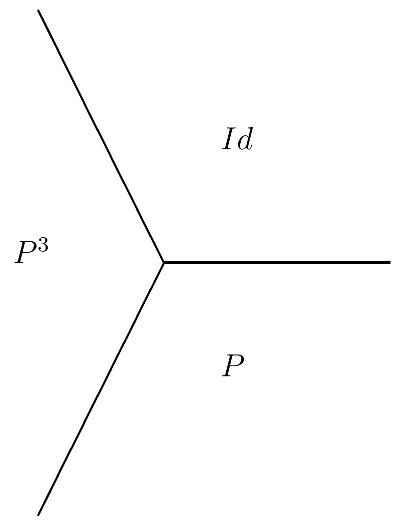

Figure 4.20: Schematic of tessellation about $(23, 1323, *)$

The cycle for this two face is

$$\begin{array}{ccccccc} (23,1323,*) & \rightarrow & (131212,12,*) & \rightarrow & (1231,12321231,*) & \rightarrow & (23,1323,*) \\ \mathbf{F}_1 \cap \mathbf{D}_1 & & \mathbf{D}_2 \cap \mathbf{E}_1 & & \mathbf{E}_2 \cap \mathbf{F}_2 & & \mathbf{F}_1 \cap \mathbf{D}_1 \end{array}.$$

The geodesics segment $[23, 1323]$ is contained in three Giraud triangle that are 2-faces of the surrounding polyhedra. These three triangles are $(13, 23, 1323)$ contained in $\mathcal{B}_A \cap P(\mathcal{B}_B)$, $(23, 1323, 1232)$ contained in $\mathcal{B}_A \cap P^3(\mathcal{B}_B)$ and $(23, 1323, 323123)$ contained in $P(\mathcal{B}_B)$ and $P^3(\mathcal{B}_B)$. To confirm the tessellation condition we only need to observe that core pieces $\mathbf{A}$, $P(\mathbf{B})$ and $P^3(\mathbf{B}_B)$ cover a three dimensional neighbourhood of $(23, 1323)$. This is clear, $\mathbf{A}$ fills in the region between the triangles $(23, 1323, 1232)$ and $(13, 23, 1323)$, the other two regions are similarly filled by the other core pieces. Then since $*$ is not contained in any of the three bisectors, the geodesic cones to $*$ over the core pieces $\mathbf{A}$, $P(\mathbf{B})$ and $P^3(\mathbf{B}_B)$ cover a neighbourhood of the geodesic cone to $*$ over the geodesic segment $[23, 1323]$.
The other two cycle relations are the essentially the same.

Tessellation about $(232121, 2321232121, *)$

In order to check this tessellation we need to show that $P^7(\Delta^0) \cap \Delta^0 = \emptyset$. We check this the same way checked the P^1, P^2, P^3 and P^5 cases for theorem 4.5.2. The

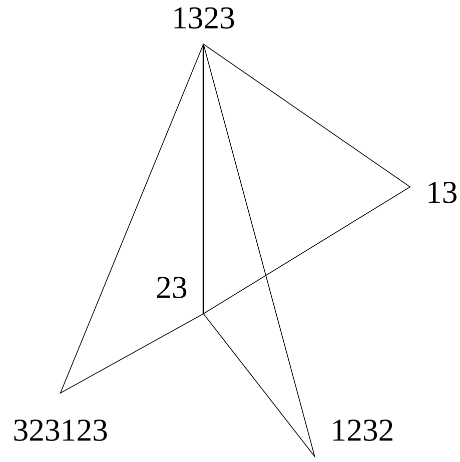

Figure 4.21: Three triangles surrounding $[23, 1323]$

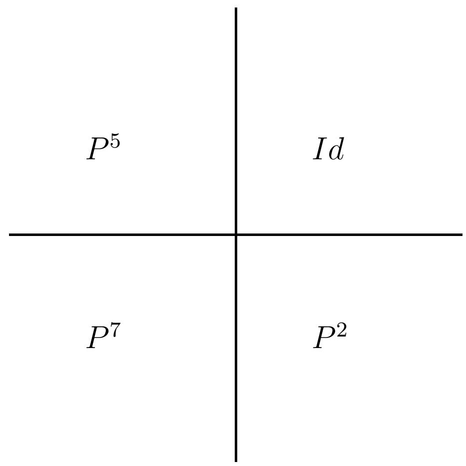

Figure 4.22: Schematic of tessellation about $(232121, 2321232121, *)$

relevant figures are included in appendix A.

Tessellation about $(2321232121, 12321323, *)$

This cycle relation is slightly different from all the others, since the cycle consists of the same side pairing (P^5) applied six times. The side pairing P^5 is an order 6 complex reflection and we chose coordinates so that its fixed line is $\{(\zeta, 0, 1)^t : \zeta \in \mathbb{C}\}$, this is also Σ_C, the complex spine of $\mathcal{B}_C$. The side pairing P^5 acts on $\mathbf{H}^2_{\mathbb{C}}$ by sending $(z_1, z_2, 1)^t$ to $(z_1, (-\omega)z_2, 1)^t$, where $\omega = (-1 + \sqrt{3})/2$. So we can partition $\mathbf{H}^2_{\mathbb{C}}$ into six segments defined as the regions where the argument of z_2

lies in one of the intervals $[0, \pi/6]$, $[\pi/6, 2\pi/3]$, ..., $[5\pi/6, 0]$. The polyhedron Δ completely covers a neighbourhood of $(2321232121, 12321323, *)$ inside the segment corresponding to $[\pi, 4\pi/6]$ and therefore $P^{5n}(\Delta)$ completely cover a neighbour of $(2321232121, 12321323, *)$ and this tessellation satisfies **cyclic**.

4.6 A presentation for Γ

Theorem 4.6.1 *Δ is a fundamental domain for the group generated by I_1 and J, furthermore the group has presentation*

$$\left\langle I_1, J \mid JI_iJ^{-1}I_{i+1},\ J^3,\ I_i^2,\ (I_iI_j)^4,\ (I_iI_jI_k)^{10},\ (I_iI_jI_kI_j)^5 \right\rangle \tag{4.8}$$

Proof: By Poincaré's polyhedron theorem the group is generated by words I_1, I_{121}, $I_{123212321}$, P, P^2, P^3, P^5. We can immediately reduce this generating set to I_1 and P using the reflection relations. Using the cycle relations we have the group relation $I_{12}^4 = Id$, conjugating this word by J and taking inverses gives us all relations of the form $I_{ij}^4 = Id$. The relation $I_{1321323132313231323123212321} = Id$ gives us all relations of the form $I_{ijkj}^5 = Id$ (taking inverses and conjugating by J as necessary). The relation $I_{ijk}^{10} = Id$ is equivalent to $(JI_1)^{30} = Id$. □

Corollary 4.6.2 *Deraux's lattice has the presentation*

$$\left\langle I_1, I_2, I_3 \ :\ I_i^2, (I_iI_j)^4, (I_iI_jI_k)^{10}, (I_iI_jI_kI_j)^5 \right\rangle \tag{4.9}$$

Proof: Deraux's lattice is an index three normal subgroup of $\langle I_1, J\rangle$, the presentation follows by Reidemeister-Schreier. □

4.7 Gauss-Bonnet formula and lattice covolume

Let $\Gamma = \langle I_1, J\rangle$. Now that we have constructed a fundamental domain we can use a form of Gauss-Bonnet theorem to calculate the volume of the orbifold $\mathbf{H}^2_{\mathbb{C}}/\Gamma$. In order to do this we have to consider the stabilizers and orbits of all n-faces of Δ.

Lemma 4.7.1 *The group generated by P^5 and I_{121} has order 72.*

PROOF: First notice that P^5 and $I_{121}P^5I_{121}$ commute and both have order 6 (this can be easily be checked using word notation or from their matrices). So the group $G = \langle P^5, I_{121}P^5I_{121}\rangle$ has order 36.

Let $H = \langle P^5\rangle$, $K = \langle I_{121}P^5I_{121}\rangle$ and $\alpha : H \to K$ be the natural homomorphism, i.e. $\alpha(g) = I_{121}gI_{121}$. Then the HNN extension of G relative to α, with stable letter I_{121}, is the group generated by P^5 and I_{121}. The order of this group is $2 \times 36 = 72$.

□

Euler orbifold characteristic for $\langle I_1, J\rangle$

Contribution to Euler orbifold characteristic from 0-faces.

Orbit representative	Stabiliser	Order	Euler
23	$\langle I_2, I_3\rangle$	8	1/8
1232	$\langle I_1, I_{232}\rangle$	10	1/10
2321232121	$\langle I_{121}, P^5\rangle$	72	1/72
$*$	$\langle P\rangle$	30	1/30
		Total	49/180

Contribution to Euler orbifold characteristic from 1-faces.

Orbit representative	Stabiliser	Order	Euler
$(13, 12)$	$\langle I_1\rangle$	2	$-1/2$
$(13, 1323)$	$\langle I_1\rangle$	2	$-1/2$
$(12, 1232)$	$\langle I_1\rangle$	2	$-1/2$
$(1323, 1232)$	$\langle I_1\rangle$	2	$-1/2$
$(232121, 2321232121)$	$\langle I_{121}\rangle$	2	$-1/2$
$(2321232121, 12321323)$	$\langle I_{2321232}, P^5\rangle$	12	$-1/12$
$(*, 12)$	Id	1	-1
$(*, 1232)$	Id	1	-1
$(*, 2321232121)$	$\langle J^5\rangle$	6	$-1/6$
		Total	$-19/4$

Contribution to Euler orbifold characteristic from 2-faces.

Orbit representative	Stabiliser	Order	Euler
$(13, 12, 23)$	$\langle J \rangle$	3	$1/3$
$(12, 23, 1232)$	Id	1	1
$(23, 1323, 1232)$	Id	1	1
$(1232, 232121, 2321232121)$	Id	1	1
$(12321231, 2321232121, 12321323)$	$\langle I_{123212321} \rangle$	2	$1/2$
$(23, 1323, *)$	Id	1	1
$(13, 1323, *)$	Id	1	1
$(13, 12, *)$	Id	1	1
$(1232, 232121, *)$	Id	1	1
$(232121, 2321232121, *)$	Id	1	1
$(2321232121, 12321323, *)$	$\langle P^5 \rangle$	6	$1/6$
		Total	9

Contribution to Euler orbifold characteristic from 3-faces.

Orbit representative	Stabiliser	Order	Euler
A	$\langle I_1 \rangle$	2	$-1/2$
B	$\langle I_{121} \rangle$	2	$-1/2$
C	$\langle I_{123212321} \rangle$	2	$-1/2$
D_1	Id	1	-1
E_1	Id	1	-1
F_1	Id	1	-1
G_1	Id	1	-1
		Total	$-11/2$

Contribution to Euler orbifold characteristic from 4-faces.

Orbit representative	Stabiliser	Order	Euler
Δ	Id	1	1
		Total	1

The Euler orbifold characteristic is

$$\frac{49}{180} - \frac{19}{4} + 9 - \frac{11}{2} + 1 = \frac{1}{45}.$$

Proposition 4.7.2 *The orbifold* $\boldsymbol{H}^2_{\mathbb{C}}/\Gamma$ *has Euler orbifold characteristic* $\chi\left(\boldsymbol{H}^2_{\mathbb{C}}/\Gamma\right) = 1/45$ *and hence*

$$\mathrm{Vol}\left(\langle \mathrm{I}_1, \mathrm{J}\rangle\right) = 8\pi^2/135.$$

Proof: We look at the orbit of an n-face and calculate the orbit of its stabilizer. These are listed in the tables above. The first column lists a representative of the orbit. The second column records the stabilizer of the orbit representative in the first column; the order of this group is clear in most cases, either the group is a cyclic or a dihedral group. The only exception is $\langle P^5, I_{121}\rangle$ which has order 72 by lemma 4.7.1. In the right hand column we record the contribution to the orbifold Euler characteristic. The covolume is $(8\pi^2/3) \times \chi(\mathbf{H}^2_{\mathbb{C}}/\Gamma)$ □

Corollary 4.7.3 *Deraux's lattice has covolume* $8\pi^2/45$.

Proof: Deraux's lattice is an index three subgroup of $\langle I_1, J\rangle$. □

Remark: In [25], Prasad and Yeung study fake projective planes and list all torsion-free cocompact arithmetic subgroups $\Gamma \subset \mathrm{PU}(2,1)$ with Euler-Poincaré characteristic $\chi(\Gamma) \leq 1$ and calculate the covolumes of the lattices. Remarkably, in section 9 of that paper, there is a lattice denoted $\mathcal{C}_2$ that is defined over the same number field as Deraux's lattice and has the same Euler-Poincaré characteristic. However the two lattices cannot be isomorphic since fake projective planes are defined over arithmetic lattices of the second type and Deraux's lattice is first type arithmetic.

Corollary 4.7.4 *The deformed triangle groups* $\Gamma(4,4,5;4)$, $\Gamma(4,5,5;4)$ *and* $\Gamma(5,5,6;4)$ *are lattices with the same presentations and covolume as* $\Gamma(4,4,4;5)$.

Proof: In chapter 2 we use the map ι to show that all these groups are identified with Deraux's lattice. □

Chapter 5

Miscellaneous results

This final chapter contains a list of many discrete deformed triangle groups and a number of partial results that may merit further investigation.

5.1 List of discrete deformed triangle groups

The following lists contain many the groups that we know to be discrete, either due to results from the previous chapter or earlier papers. Some of the rows are grouped together by horizontal lines, this indicates that these groups can be identified by the ι maps from chapter 2. The first column records the value $\mathbf{K}$, introduced in chapter 2. Recall from remark 2.4, for the group to be non-degenerate we require that $\mathbf{K}+4<0$. For $\Gamma(18,18,18;18)$, $\mathbf{K}+4=-0.113340801\ldots$, it would be interesting to know if this group has the smallest value of $\mathbf{K}+4$ for a discrete group. Similarly $\mathbf{K}+4$ is relatively small for the four known deformed triangle group lattices. The other columns records the order of the respective word, with parabolic words denoted be an ∞ and loxodromic words by lox.

Definition 5.1.1 *In the tables below we denote a triple* (p,q,r) ***(P)*** *if*

$$4+|\rho\sigma\tau|-|\rho|^2-|\sigma|^2-|\tau|^2=0.$$

This is equivalent to the condition $r_1^2+r_2^2+r_3^2=1+2r_1r_2r_3$ *which was introduced in section 12 of [26]. Whether or not a group satisfies* $(\boldsymbol{P})$ *is not invariant under* ι_i.

5.1.1 List of discrete deformed triangle groups without short parabolic words

$\mathbf{K}+4$	23	31	12	1323	3212	2131	123	notes
-0.113340801	18	18	18	18	18	18	9	Livné group
-0.118033989	3	3	5	5	5	5	15	Lattice
-0.118033989	3	5	5	3	3	10	15	Lattice
-0.118033989	5	5	10	3	5	5	15	Lattice
-0.123489801	3	3	4	7	7	7	42	Lattice
-0.123489801	3	4	7	3	3	7	42	Lattice
-0.123489801	3	3	7	4	4	4	42	Lattice
-0.123489801	4	7	7	3	3	14	42	Lattice
-0.123489801	7	7	14	4	7	7	42	Lattice
-0.232050809	12	12	12	12	12	12	12	Livné group
-0.309016995	4	4	4	5	5	5	10	Lattice
-0.309016995	4	4	5	4	5	5	10	Lattice
-0.309016995	4	5	5	4	4	6	10	Lattice
-0.309016995	5	5	6	4	5	5	10	Lattice
-0.309016991	10	10	10	10	10	10	5	Livné group
-0.358440714	9	9	9	9	9	9	18	Livné group
-0.414213562	8	8	8	8	8	8	8	Livné group
-0.427050985	5	5	5	5	5	5	10	Lattice+Livné
-0.427050985	3	4	6	6	6	lox	lox	
-0.427050985	3	6	6	4	4	lox	lox	(**P**)
-0.469500539	7	7	7	7	7	7	14	Livné group
-0.618033989	4	5	5	5	5	10	lox	
-0.618033989	5	5	10	4	10	10	lox	
-0.618033989	5	10	10	5	5	lox	lox	(**P**)
-0.618033989	6	7	7	7	7	14	lox	
-0.618033989	7	7	14	6	lox	lox	lox	
$1/2-\cos(2\pi/n)$	3	3	4	n	n	n	lox	

$1/2 - \cos(2\pi/n)$	3	3	n	4	4	3	lox	n =7,8,9,10,11,
$1/2 - \cos(2\pi/n)$	3	4	n	3	3	n	lox	12,14,16,18,24,30
$1/2 - \cos(2\pi/n)$	4	n	n	3	3	lox	lox	
$-\cos(2\pi/n)$	3	3	6	n	n	n	lox	
$-\cos(2\pi/n)$	3	3	n	6	6	6	lox	n =5,6,7,8,10,12,18
$-\cos(2\pi/n)$	3	6	n	3	3	lox	lox	
$-\cos(2\pi/n)$	4	4	4	n	n	n	lox	
$-\cos(2\pi/n)$	4	4	n	4	n	n	lox	n = 5,6,7,8,10,12,18
$-\cos(2\pi/n)$	4	n	n	4	4	lox	lox	
-1	4	6	6	6	6	lox	lox	
-1.5	3	6	6	6	6	lox	lox	(**P**)

5.1.2 List of discrete deformed triangle groups with short parabolic words

$\mathbf{K}+4$	23	31	12	1323	3212	2131	123	notes
-0.5	4	4	4	6	6	6	lox	
-0.5	4	4	6	4	6	6	lox	
-0.5	4	6	6	4	4	∞	lox	
-0.5	6	6	∞	4	∞	∞	lox	(**P**)
-0.5	6	∞	∞	6	6	lox	lox	
-0.5	4	4	∞	3	6	6	lox	(**P**)
-0.5	4	6	∞	3	4	lox	lox	
-0.5	3	4	6	4	4	∞	lox	
-0.5	3	4	4	6	6	∞	lox	
-0.5	6	6	6	6	6	6	∞	Livné Group
-0.809016995	4	5	5	6	6	∞	lox	
-0.809016995	5	5	∞	4	lox	lox	lox	(**P**)
-0.809016995	4	6	6	5	5	lox	lox	
-0.809016995	4	5	6	5	6	lox	lox	
-1	4	4	6	6	∞	∞	lox	
-1	4	6	∞	4	6	lox	lox	
-1	3	3	∞	6	6	6	lox	(**P**)
-1	3	6	∞	3	3	3	lox	
-1	3	3	6	∞	∞	∞	lox	
-1	4	4	4	∞	∞	∞	lox	
-1	6	6	6	∞	∞	∞	lox	
-1	6	6	∞	6	lox	lox	lox	(**P**)
-2	4	4	∞	∞	lox	lox	lox	(**P**)

5.1.3 List of possible discrete deformed triangle groups

We now list a few groups that we have not been able to determine whether or not they are discrete, obviously this list is not exhaustive; we restrict ourselves to interesting groups (e.g. groups where all words of length 4 are finite order regular elliptic or non-standard groups).

$\mathbf{K}+4$	23	31	12	1323	3212	2131	123	notes
-0.309016992	3	4	4	5	5	10	lox	
-0.309016992	3	4	5	4	4	10	lox	
-0.309016992	4	4	10	3	5	5	lox	
-0.309016992	4	5	10	3	4	10	lox	
-0.309016992	5	10	10	4	4	lox	lox	(**P**)
-0.400968868	4	7	14	3	7/2	lox	lox	
-0.400968868	3	4	7	7/2	7/2	14	lox	Non-standard
-0.809016995	4	4	5	6	10	10	lox	
-0.809016995	4	4	6	5	10	10	lox	
-0.809016995	4	5	10	4	6	lox	lox	
-0.809016995	4	6	10	4	5	lox	lox	
-0.809016995	5	5	6	6	10	10	lox	
-0.809016995	5	6	10	5	10	lox	lox	
-0.809016995	5	5	6	6	10	10	lox	
-0.939692620	4	9	18	4	9/2	lox	lox	
-0.939692620	4	4	9	9/2	18	18	lox	Non-standard
-0.978147601	4	5	30	4	15/2	lox	lox	
-0.978147601	4	4	5	15/2	30	30	lox	Non-standard
-1.478147601	4	10	30	6	15/2	lox	lox	
-1.478147601	4	6	10	15/2	30	lox	lox	Non-standard

5.2 Deformed triangle subgroups of $\mathbf{PU}\,(2,1;\mathcal{O}_7)$

The following is the result of an unsuccessful attempt to produce a presentation of $\Gamma = \mathbf{PU}\,(2,1;\mathcal{O}_7)$ (where $\mathcal{O}_7 = \mathbb{Z}\left[\left(1+\sqrt{-7}\right)/2\right]$) by looking at some of its subgroups that are deformed triangle groups. We began by trying to construct a fundamental domain for Γ following the method used by Falbel and Parker for $\mathbf{PU}\left(2,1;\mathbb{Z}\left[\left(1+\sqrt{-3}\right)/2\right]\right)$ [9]. This lead to a domain that was much to complicated to understand.

$\Gamma(3,3,4;\infty)$

Let $k=(1+\sqrt{-7})/2$, then we have the following representation for $(3,3,4;\infty)$,

$$I_1=\begin{pmatrix} -1 & -k & 1\\ 0 & 1 & -\overline{k}\\ 0 & 0 & -1\end{pmatrix},\quad I_2=\begin{pmatrix} -1 & 0 & 0\\ -k & 1 & 0\\ 1 & -\overline{k} & -1\end{pmatrix},$$

$$I_3=\begin{pmatrix} 0 & 0 & 1\\ 0 & -1 & 0\\ 1 & 0 & 0\end{pmatrix}.$$

$\Gamma(3,4,\infty;\infty)$

Let $k=(1+\sqrt{-7})/2$, then we have the following representation for $(3,4,\infty;\infty)$,

$$I_1=\begin{pmatrix} -1 & 2 & 2\\ 0 & 1 & 2\\ 0 & 0 & -1\end{pmatrix},\quad I_2=\begin{pmatrix} -1 & 0 & 0\\ -\overline{k} & 1 & 0\\ 1 & -k & -1\end{pmatrix},$$

$$I_3=\begin{pmatrix} 0 & 0 & 1\\ 0 & -1 & 0\\ 1 & 0 & 0\end{pmatrix}.$$

$\Gamma(4,4,4;\infty)$

Let $k=(1+\sqrt{-7})/2$, then we have the following representation for $(4,4,4;\infty)$,

$$I_1=\begin{pmatrix} -1 & -1-\overline{k} & 2 \\ 0 & 1 & -1-k \\ & 0 & -1 \end{pmatrix},\quad I_2=\begin{pmatrix} -1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 2 & -1 \end{pmatrix},$$

$$I_3=\begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Each of the above representation preserve the hermitian form

$$H=\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

and the fixed point of the parabolic isometry I_{1323} is $(1,0,0)$, the point at infinity.

$\mathbf{PU}\left(2,1;\mathbb{Z}\left[\left(1+\sqrt{-7}\right)/2\right]\right)$

Following Swan's convention for Bianchi groups in [34] we define the following matrices in $\mathbf{PU}\left(2,1;\mathbb{Z}\left[\left(1+\sqrt{-7}\right)/2\right]\right)$.

$$J=\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},\quad T=\begin{pmatrix} 1 & -1 & -\overline{k} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix},\quad U=\begin{pmatrix} 1 & -\overline{k} & -1 \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix},$$

$$A=\begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix},\quad Z=\begin{pmatrix} 1 & 0 & \overline{k}-k \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Notice that $Z=U^{-1}T^{-1}UT$ and $J=AUAU^{-1}AUA$. We can express the above groups in terms of these matrices as follows,

$$(3,3,4;\infty)=\langle JT^{-1}U,\ JAUA,\ A\rangle,$$

$$(3,4,\infty;\infty)=\langle T^{-1}JT,\ T^{-1}AUJAT,\ A\rangle,$$

$$(4,4,4;\infty)=\langle TUJ,\ ATJT^{-1}A,\ A\rangle.$$

We believe that U, T and A are a generating set for $\mathbf{PU}\left(2,1;\mathbb{Z}\left[\left(1+\sqrt{-7}\right)/2\right]\right)$, unfortunately the method used in [9] leads to insurmountable difficulties when we attempt to construct a fundamental domain for the group.

5.3 Higher order reflection groups

Recall from chapter 1 that a reflection in a $\mathbb{C}$-line need not necessarily have order 2. Given a triangle of $\mathbb{C}$-lines we can also consider the group generated by higher order complex reflections. We will follow Parker and Paupert's convention in [24].

We define higher order deformed triangle groups as a representation of the abstract group

$$\langle R_1, R_2, R_3 \mid R_i^k, \left(R_2R_3^{-1}\right)^p, \left(R_3R_1^{-1}\right)^q, \left(R_1R_2^{-1}\right)^r \rangle$$

into $\mathbf{PU}(2,1)$. Let k be the order of the reflection then $\psi = 2\pi/k$ is the angle of rotation.

Now we define the matrices R_1, R_2, R_3,

$$R_1 = e^{-i\psi/3}\begin{pmatrix} e^{i\psi} & \rho & -\overline{\tau} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad R_2 = e^{-i\psi/3}\begin{pmatrix} 1 & 0 & 0 \\ -e^{i\psi}\overline{\rho} & e^{i\psi} & \sigma \\ 0 & 0 & 1 \end{pmatrix},$$
$$R_3 = e^{-i\psi/3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ e^{i\psi}\tau & -e^{i\psi}\overline{\sigma} & e^{i\psi} \end{pmatrix}.$$

preserving the Hermitian form

$$H = \begin{pmatrix} 2-2\mathrm{Re}\left(e^{i\psi}\right) & \rho(e^{-i\psi}-1) & \overline{\tau}(1-e^{-i\psi}) \\ \overline{\rho}(e^{i\psi}-1) & 2-2\mathrm{Re}\left(e^{i\psi}\right) & \sigma(e^{-i\psi}-1) \\ \tau(1-e^{i\psi}) & \overline{\sigma}(e^{i\psi}-1) & 2-2\mathrm{Re}\left(e^{i\psi}\right) \end{pmatrix}$$

In order for the group generated by R_1, R_2 and R_3 to be a complex hyperbolic triangle group, we need to ensure that this Hermitian form has signature $(2,1)$.

It follows from a straightforward calculation that $\mathrm{tr}(R_1R_2) = e^{i\psi/3}(2-|\rho|^2) + e^{-2i\psi/3}$. There are equivalent expressions for the traces of R_2R_3 and R_3R_1. We can also calculate the trace of $R_1R_2^{-1}$ as $1+|\rho|^2+2\cos(\psi)$, again there are equivalent expressions for the traces of $R_2R_3^{-1}$ and $R_3R_1^{-1}$.

We now consider the triple product $R_1R_2R_3$.

$$R_{123}=\begin{pmatrix} 1-|\rho|^2+\rho\sigma\tau-|\tau|^2 & \rho(1-|\sigma|^2)+\overline{\sigma\tau} & \rho\sigma-\overline{\tau} \\ \sigma\tau-\overline{\rho} & 1-|\sigma|^2 & \sigma \\ \tau & -\overline{\sigma} & 1 \end{pmatrix}$$

Remarkably the factor of $e^{i\psi/3}$ vanishes.

In the case of order 2 reflections we were particularly interested in the cases where I_{123} was regular elliptic. Since the factor of $e^{i\psi/3}$ vanishes from the triple product and we have $R_{123}=I_{123}$ for any value of ψ. Then it follows that if we can find values for ρ, σ and τ where the order 2 triple product I_{123} is regular elliptic then when we increase the order R_{123} remains regular elliptic.

5.3.1 Higher order analogues of $\Gamma(3,3,4;7)$ and $\Gamma(3,3,5;5)$

Recall from section 3.1 if we set

$$\sigma=u^5,\quad \tau=u^5,\quad \rho=u+u^2+u^4.$$

where $u=e^{2i\pi/7}$, then the resulting presentation for the group generated by order two reflections was $\Gamma(3,3,4;7)$, and in particular I_{123} has order 42. Then in the above presentation R_{123} will also be regular elliptic and have order 42. Now we check to see in there are any rational values of ψ such that $R_iR_j^{-1}$ will be regular elliptic. Clearly $\psi=\pi$ will be such a value since this corresponds to the order 2 reflection group. We use the trace formula described above, namely:

$$\begin{aligned}\mathrm{tr}(R_1R_2^{-1})&=1+|\rho|^2+2\cos(\psi),\\ \mathrm{tr}(R_2R_3^{-1})&=1+|\sigma|^2+2\cos(\psi),\\ \mathrm{tr}(R_3R_1^{-1})&=1+|\tau|^2+2\cos(\psi).\end{aligned}$$

Then using lemma 2.3.4 we can check values of ψ where these words are regular elliptic and have finite order. Since $|\rho|^2=2$ and $|\sigma|^2=|\tau|^2=1$ these equations become

$$\mathrm{tr}(R_1R_2^{-1}) = 3 + 2\cos(\psi),$$
$$\mathrm{tr}(R_2R_3^{-1}) = 2 + 2\cos(\psi),$$
$$\mathrm{tr}(R_3R_1^{-1}) = 2 + 2\cos(\psi).$$

For the words to be elliptic, the trace has to lie in the interval $(-1, 3)$, so clearly the only permitted values of ψ are $2\pi/2$ and $2\pi/3$, corresponding to the groups generated by order 2 and order 3 reflections respectively. It follows from a straight forward calculation that in the order 3 reflection group R_1R_2 have order 12, R_2R_3 and R_3R_1 has order 6. Similarly $R_1R_2^{-1}$ has order 6, $R_2R_3^{-1}$ and $R_3R_1^{-1}$ have order 4. There is clearly a question as to whether we refer this group as a deformed $(6, 6, 12)$ group or a $(4, 4, 6)$ group. We sidestep this issue for the moment by using the dual notation $\Gamma^+(6, 6, 12)$ and $\Gamma^-(4, 4, 6)$ to describe deformed triangle groups relative to R_iR_j or $R_iR_j^{-1}$ respectively.

The word $R_1R_3R_2^{-1}R_3^{-1}$ has trace $3+2\mathrm{Re}\,(\rho\sigma\tau\overline{\omega}) = 0.683356440\ldots$, in particular it is regular elliptic. Using lemma 2.3.4 this word does not appear to be finite order, if this is the case then the group is not discrete.

Let $u = e^{2i\pi/5}$, if we set

$$\sigma = u^2, \quad \tau = u^2, \quad \rho = -1 - u^4.$$

then the resulting presentation for the group generated by order two reflections was $\Gamma(3, 3, 5; 5)$, and in particular I_{123} has order 15. Following the same approach as above, we only get a group for order 3 reflections, namely a deformed $\Gamma^+(6, 6, 30)$ / $\Gamma^-(4, 4, 10)$ group.

5.3.2 More general higher order reflection groups

For more general triangle groups, we use the $\Gamma^-(p, q, r)$, convention, i.e. we use $R_iR_j^{-1}$ to fix three of the four variables parametrising the space of hyperbolic triangles, rather than R_iR_j. We use this convention because the trace of $R_iR_j^{-1}$ is real, so it easier to ensure that the corresponding word is regular elliptic with finite order.

If we choose three real numbers $|\rho|$, $|\sigma|$, $|\tau|$ such that

$$|\sigma|^2 = 2\cos(2\pi/p) - 2\cos(\psi),$$
$$|\tau|^2 = 2\cos(2\pi/q) - 2\cos(\psi),$$
$$|\rho|^2 = 2\cos(2\pi/r) - 2\cos(\psi).$$

this forces $R_2R_3^{-1}$, $R_3R_1^{-1}$ and $R_1R_2^{-1}$ to have order p, q, r respectively. Note that in this convention $R_iR_{i+1}^{-1}$ cannot have order less than the generating complex reflection. Then if we set

$$t = \arccos\left(\frac{2\cos(2\pi/n) - 2\cos(\psi) - |\rho|^2 - |\sigma|^2|\tau|^2}{2|\rho\sigma\tau|}\right). \tag{5.1}$$

and let $\rho = |\rho|e^{it/3}$, $\sigma = |\sigma|e^{it/3}$ and $\tau = |\tau|e^{it/3}$, this forces $R_2R_3R_1^{-1}R_3^{-1}$ to have order n. We denote the group generated by these matrices as $\Gamma^-(p_k, q_k, r_k; n)$. As before, the superscript minus sign records the fact we are using the order of $R_iR_{i+1}^{-1}$ to determine $|\rho|$, $|\sigma|$ and $|\tau|$ and the subscript k records the order of the complex reflections. When we use the order of R_iR_j to determine the group we will denote it by $\Gamma^+(p_k, q_k, r_k; n)$.

Lemma 5.3.1 *In the presentation given above the following words all have real trace*

$$\begin{gathered} R_1R_2^{-1},\ R_2R_3^{-1},\ R_3R_1^{-1}, \\ R_1R_2R_3^{-1}R_2^{-1},\ R_2R_3R_1^{-1}R_3^{-1},\ R_3R_1R_2^{-1}R_1^{-1}, \\ R_1R_2R_1^{-1}R_2^{-1},\ R_2R_3R_2^{-1}R_3^{-1},\ R_3R_1R_3^{-1}R_1^{-1}. \end{gathered} \tag{5.2}$$

Proof: Follows from straightforward multiplication of the matrices. □

Since these words have real trace, we can use a computer program similar to the one from section 2.6.

The computer program checks the traces to find groups where these words are finite order elliptic and the word $R_1R_2R_3$ is also elliptic. The trace of $R_1R_2R_3$ is not real, so it is not straightforward to determine whether it is finite order of not. As in the order 2 case, we discard any groups of the form $(p_x, p_x, p_x; n)$ since these are currently in the process of being analysed by Paupert and Parker in [24]. In addition we can't use the Jørgensen test from the chapter 2 case since all higher order reflection groups seem to never satisfy these inequalities so they are of no use in determining discreteness. However we can use the fact that the above words have real trace to check if they are finite order elliptic.

5.4 Order 3 reflections

If we limit to groups with $p_3, q_3, r_3, n < 100$, then the program produces the following list of groups where all the words (5.2) are finite order elliptic and $R_1R_2R_3$ elliptic (since $k = 3$ for all these groups we omit the subscript). $\Gamma^-(4,4,6;n)$ for all $4 \leq n < 100$, $\Gamma^-(4,4,10;4)$, $\Gamma^-(4,4,10;5)$, $\Gamma^-(4,4,10;6)$, $\Gamma^-(4,4,10;7)$, $\Gamma^-(4,4,10;8)$, $\Gamma^-(4,4,10;9)$, $\Gamma^-(4,4,10;10)$, $\Gamma^-(4,4,10;11)$, $\Gamma^-(4,6,6;4)$, $\Gamma^-(4,6,6;5)$, $\Gamma^-(4,6,10;4)$, $\Gamma^-(4,10,10;3)$, $\Gamma^-(6,6,10;4)$, $\Gamma^-(6,6,10;6)$, $\Gamma^-(6,10,10;4)$, $\Gamma^-(6,10,10;5)$. These groups are somehow the order 3 reflection analogue of the lattice candidates from chapter 2.

We now check by hand the eigenvalues of $R_1R_2R_3$ for these groups to determine when the word has finite order and we get the following list of lattice candidates.

- In $\Gamma^-(4,4,6;4)$, $R_1R_2R_3$ has order 8 and $\rho\sigma\tau = 1+i$.
- In $\Gamma^-(4,4,6;6)$, $R_1R_2R_3$ has order 7 and $\rho\sigma\tau = (1+\sqrt{-7})/2$.
- In $\Gamma^-(4,4,6;10)$, $R_1R_2R_3$ has order 5 and $\rho\sigma\tau = 1+\omega\phi$.
- In $\Gamma^-(4,4,10;4)$, $R_1R_2R_3$ has order 10 and $\rho\sigma\tau = (\phi^2 + i\sqrt{\phi^2+1})/2$.
- In $\Gamma^-(4,4,10;6)$, $R_1R_2R_3$ has order 5 and $\rho\sigma\tau = -\overline{\omega}\phi$.
- In $\Gamma^-(4,4,10;10)$, $R_1R_2R_3$ has order 15 and $\rho\sigma\tau = 1/2 + i\phi\sqrt{\phi^2+1}$.
- In $\Gamma^-(4,6,6;4)$, $R_1R_2R_3$ has order 7 and $\rho\sigma\tau = 1+(1+\sqrt{-7})/2$.
- In $\Gamma^-(4,6,10;4)$, $R_1R_2R_3$ has order 5 and $\rho\sigma\tau = 1-\overline{\omega}\phi$.
- In $\Gamma^-(6,6,10;6)$, $R_1R_2R_3$ has order 10 and $\rho\sigma\tau = 1-\overline{\omega}\phi^2$.
- In $\Gamma^-(6,10,10;4)$, $R_1R_2R_3$ has order 5 and $\rho\sigma\tau = \phi^2-\overline{\omega}\phi$.

We also check the order of the words R_iR_{i+1} for these groups, somewhat surprisingly all these words are regular elliptic of finite order.

	ord(R_2R_3)	ord(R_3R_1)	ord(R_1R_3)
$\Gamma^-(4,4,6;4)$	6	6	12
$\Gamma^-(4,4,6;6)$	6	6	12
$\Gamma^-(4,4,6;10)$	6	6	12
$\Gamma^-(4,4,10;4)$	6	6	30
$\Gamma^-(4,4,10;6)$	6	6	30
$\Gamma^-(4,4,10;10)$	6	6	30
$\Gamma^-(4,6,6;4)$	6	12	12
$\Gamma^-(4,6,10;4)$	6	12	30
$\Gamma^-(6,6,10;6)$	12	12	30
$\Gamma^-(6,10,10;4)$	12	30	30

This allows us to translate a deformed $\Gamma^-(p,q,r)$ group into a deformed $\Gamma^+(p,q,r)$ group.

5.5 Order 5 reflections

As with the order 3 reflections, we restrict to $p,q,r,n < 100$ then the program produces the following list of groups with satisfying criteria. We omit the subscript 5.

- $\Gamma^-(6,6,10;6)$,
- $\Gamma^-(6,10,10;n)$ for all $6 \leq n < 100$.

As in section 5.4 we check the eigenvalues of $R_1R_2R_3$ to find those groups where is has finite order. The group $\Gamma^-(6,6,10;6)$ is degenerate, but appeared due to a rounding error, after removing it, we get the following list,

- $\Gamma^-(6,10,10;6)$, $R_1R_2R_3$ has order 15 and $\rho\sigma\tau = 1 + i\sqrt{3-\phi}/2\phi$.
- $\Gamma^-(6,10,10;10)$, $R_1R_2R_3$ has order 10 and $\rho\sigma\tau = (1 + i\sqrt{4\phi^2-1})/2\phi^2$.

Again checking the order of R_iR_{i+1} in these groups we get

	ord(R_2R_3)	ord(R_3R_1)	ord(R_1R_3)
$\Gamma^-(6,10,10;6)$	10	30	30
$\Gamma^-(6,10,10;10)$	10	30	30

5.6 Other order reflections

Using the computer program we checked for reflections of up to order 50 and $3 \leq p,q,r,n \leq 100$. There seems to be no other groups that satisfy the conditions of criteria of the programme. These 12 groups seem to be good candidates for lattices amongst higher order reflection triangle groups.

Appendix A

Projection figures

Theorem 4.5.2 For $n \in \{0, \pm 1, \pm 2, \pm 3, \pm 5\}$ $P^n(\Delta^0)$ lies entirely on the good half of $\mathbf{H}^2_{\mathbb{C}}$ with respect to $\mathcal{B}_A$, $\mathcal{B}_B$ and $\mathcal{B}_C$.

PROOF:[or theorem 4.5.2] By lemmas 4.3.5 to 4.3.8, we know that the 0 and 1-skeletons on $P^n(\Delta)$ lie in the good half of $\mathbf{H}^2_{\mathbb{C}}$ with respect to $\mathcal{B}_A$, $\mathcal{B}_B$ and $\mathcal{B}_C$ (or in the bisectors themselves). The figures below show, via the same argument that we used to prove theorem 4.5.2 that the 2-skeleton of the core faces and the cone faces lie on the good side $\mathbf{H}^2_{\mathbb{C}}$ (or, again in the bisectors).

Now we show that the core faces of $P^n(\Delta)$ (i.e. $P^n(\mathbf{A})$, $P^n(\mathbf{B})$ and $P^n(\mathbf{C})$) lie entirely on the good side of $\mathbf{H}^2_{\mathbb{C}}$ with respect to $\mathcal{B}_A$, $\mathcal{B}_B$ and $\mathcal{B}_C$. These core faces are themselves contained bisectors and a quick analysis of complex spines shows that these bisectors are coequidistant with $\mathcal{B}_A$, $\mathcal{B}_B$ and $\mathcal{B}_C$. This means that the intersection of the bisectors containing the core faces and $\mathcal{B}_A$, $\mathcal{B}_B$ and $\mathcal{B}_C$ is a smooth 2-dimensional disc. We already know that the 2-dimensional boundary of the core faces lies on the good side of the bisectors, from which it follows the entire core face must be contained in the good side of $\mathbf{H}^2_{\mathbb{C}}$ with respect to $\mathcal{B}_A$, $\mathcal{B}_B$ and $\mathcal{B}_C$ (or in the bisectors themselves).

Since the core faces lie on the good side of the bisectors $\mathcal{B}_A$ ($\mathcal{B}_B$ and $\mathcal{B}_C$), the projection of the core faces onto the complex spines Σ_A, (Σ_B and Σ_C) lie on the same side of σ_A (σ_B and σ_C) as the projection of $*$. Let p be a point in the interior of a core face then since the core faces lie on the good side of the bisector the projection of p lies on the good side of the real spine. In order to ensure that the geodesic segment $[p,*]$ does not intersect the bisector, we only need to check that the projection of the point does not lie in either of the bad regions of the complex plane as described in the proof of theorem 4.5.2. Naively, it is obvious from the figures that this is the case however we make the argument rigorous as follows. The core faces are contained in bisectors which are foliated by $\mathbb{C}$-lines, we can restrict this foliation to the core face to produce a foliation of the core faces by polygons bounded by hypercycles contained in $\mathbb{C}$-lines. Since these polygons are contained in $\mathbb{C}$-lines, they are mapped diffeomorphically onto polygons bounded by arcs of circles whose vertices are contained in the projection of the 1-skeleton of the core face. Such a polygon cannot intersect either of the bad regions as this would require one of the boundary circles to be so large that it would pass outside the $\mathbb{C}$-line.

So the projection of the geodesic segment $[p,*]$ cannot intersect the real spine, therefore the geodesic segment itself cannot intersect the bisector and the interior of Δ must lie entirely on the good side of $\mathbf{H}^2_{\mathbb{C}}$ with respect to $\mathcal{B}_A$, $\mathcal{B}_B$ and $\mathcal{B}_C$. □

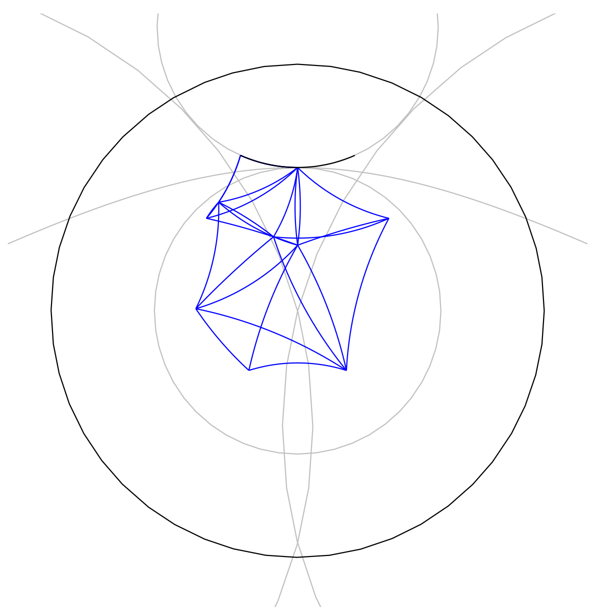

Figure A.1: Projection of the 1-skeleton of $P^{-1}(\Delta)$ onto Σ_A

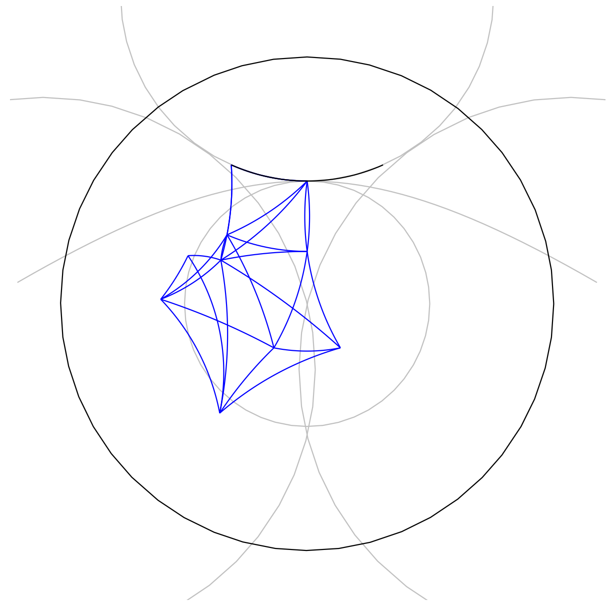

Figure A.2: Projection of the 1-skeleton of $P^{-1}(\Delta)$ onto Σ_B

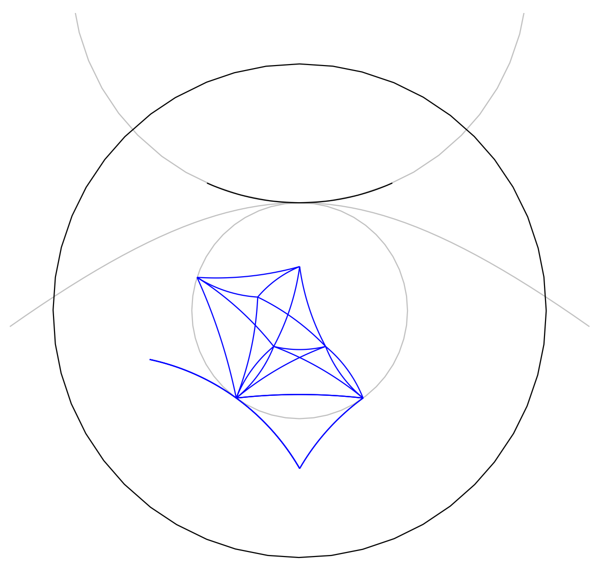

Figure A.3: Projection of the 1-skeleton of $P^{-1}(\Delta)$ onto Σ_C

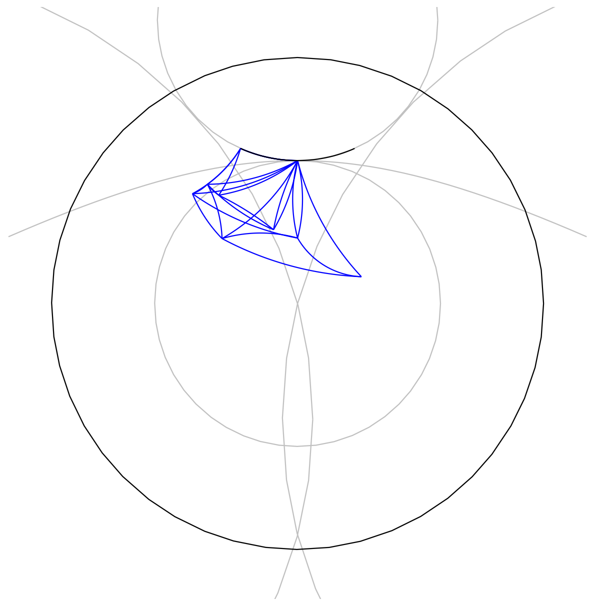

Figure A.4: Projection of the 1-skeleton of $P^{-2}(\Delta)$ onto Σ_A

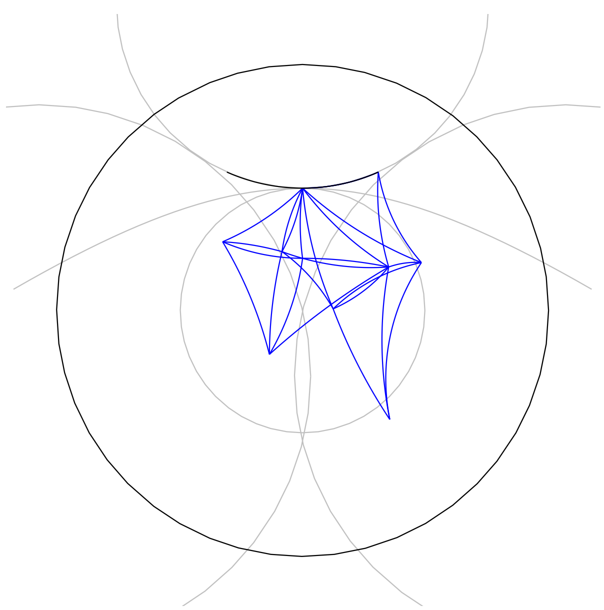

Figure A.5: Projection of the 1-skeleton of $P^{-2}(\Delta)$ onto Σ_B

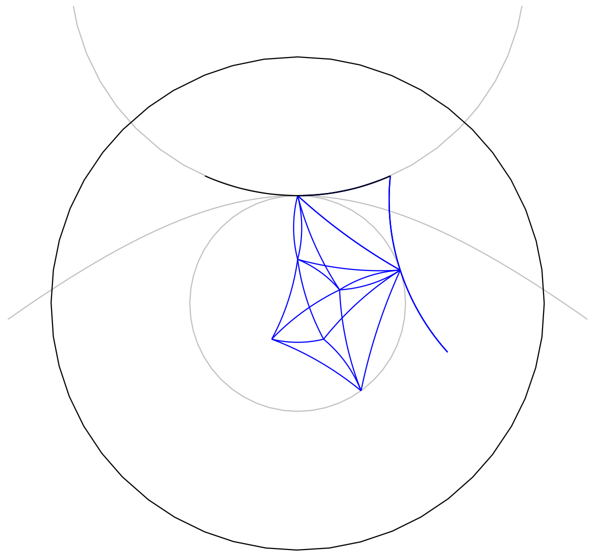

Figure A.6: Projection of the 1-skeleton of $P^{-2}(\Delta)$ onto Σ_C

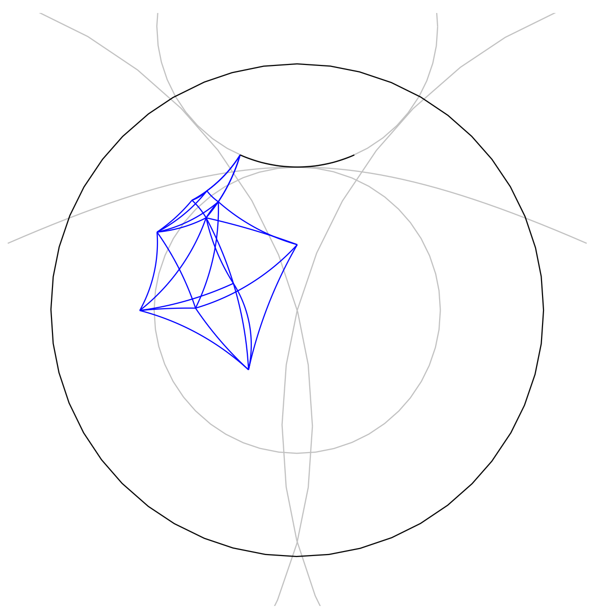

Figure A.7: Projection of the 1-skeleton of $P^{-3}(\Delta)$ onto Σ_A

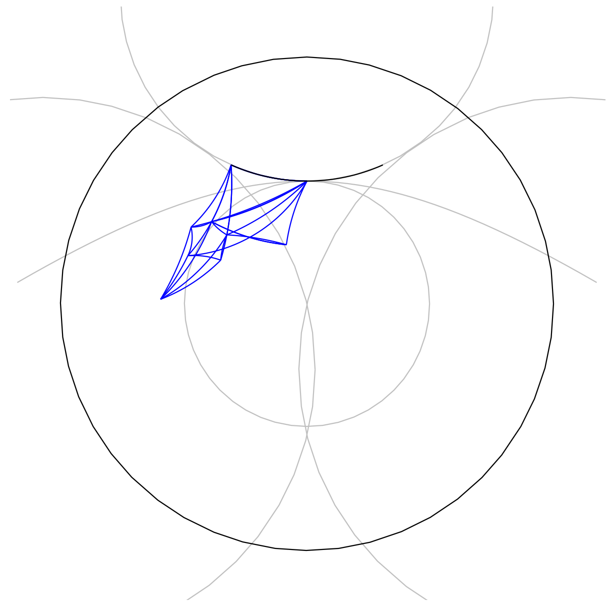

Figure A.8: Projection of the 1-skeleton of $P^{-3}(\Delta)$ onto Σ_B

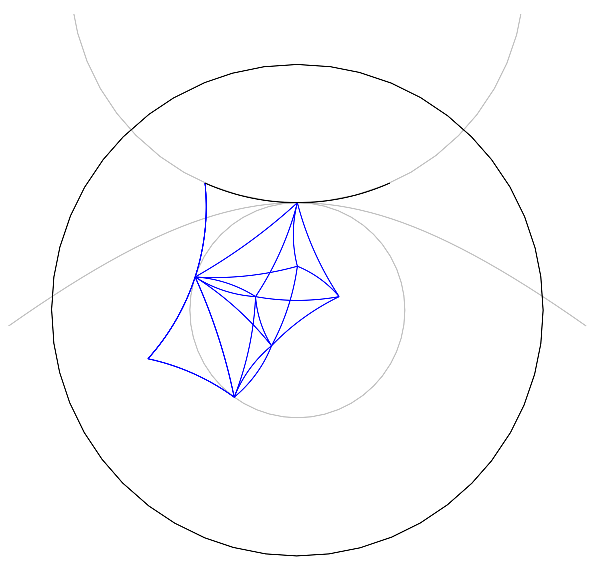

Figure A.9: Projection of the 1-skeleton of $P^{-3}(\Delta)$ onto Σ_C

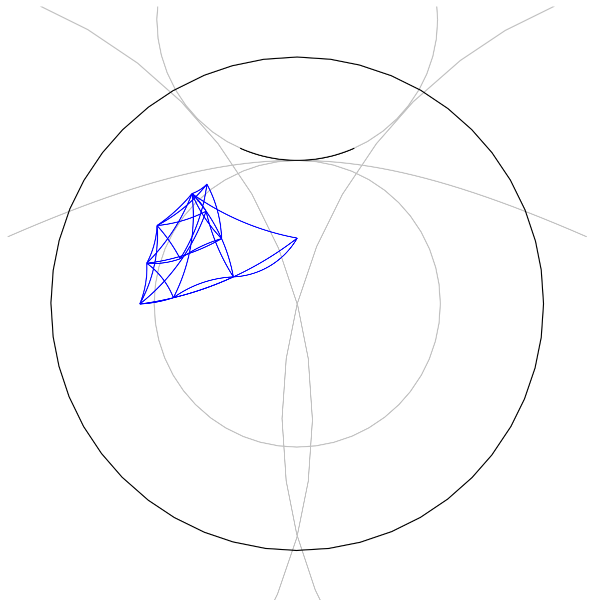

Figure A.10: Projection of the 1-skeleton of $P^{-5}(\Delta)$ onto Σ_A

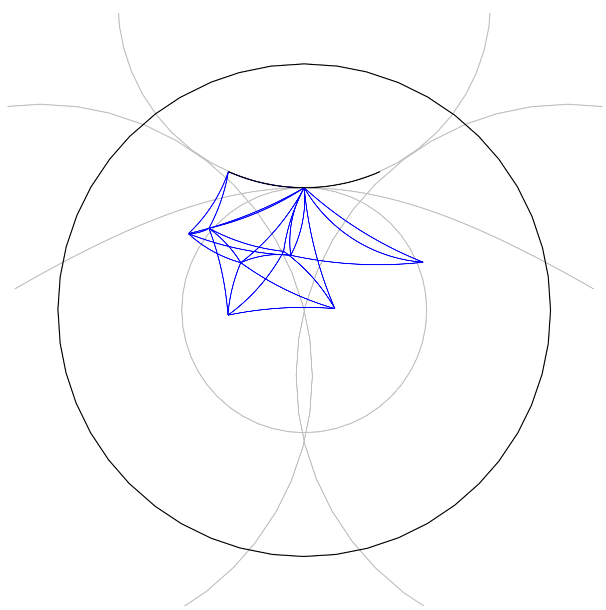

Figure A.11: Projection of the 1-skeleton of $P^{-5}(\Delta)$ onto Σ_B

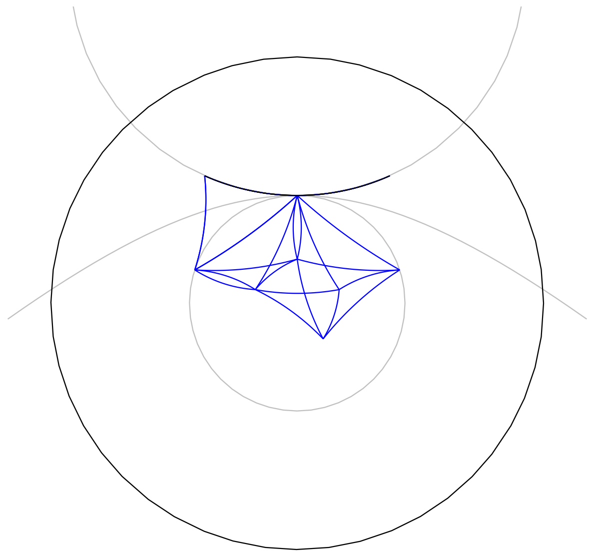

Figure A.12: Projection of the 1-skeleton of $P^{-5}(\Delta)$ onto Σ_C

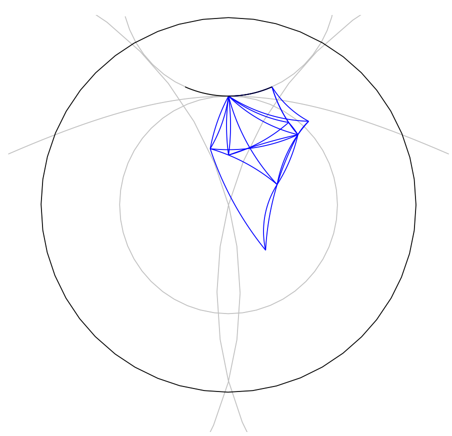

Figure A.13: Projection of the 1-skeleton of $P^{1}(\Delta)$ onto Σ_A

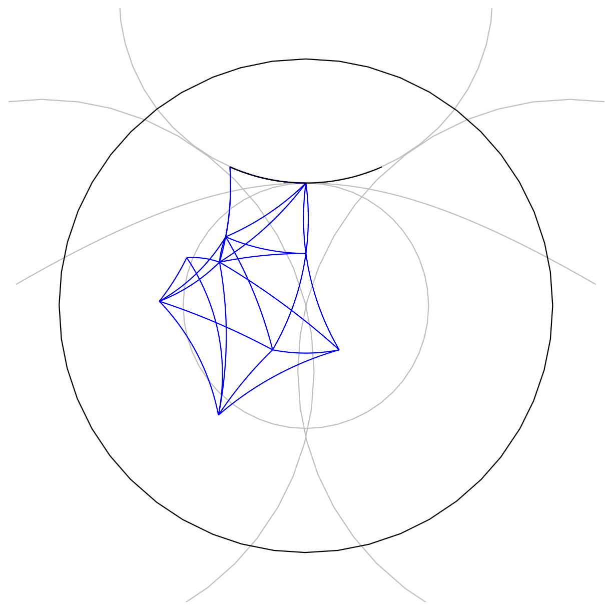

Figure A.14: Projection of the 1-skeleton of $P^1(\Delta)$ onto Σ_B

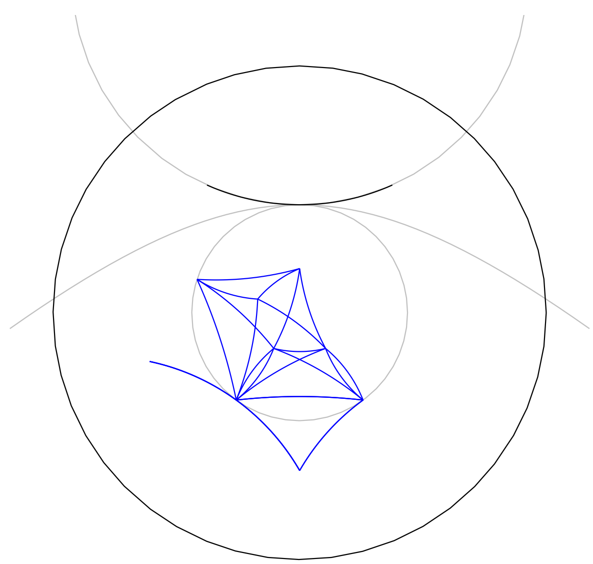

Figure A.15: Projection of the 1-skeleton of $P^1(\Delta)$ onto Σ_C

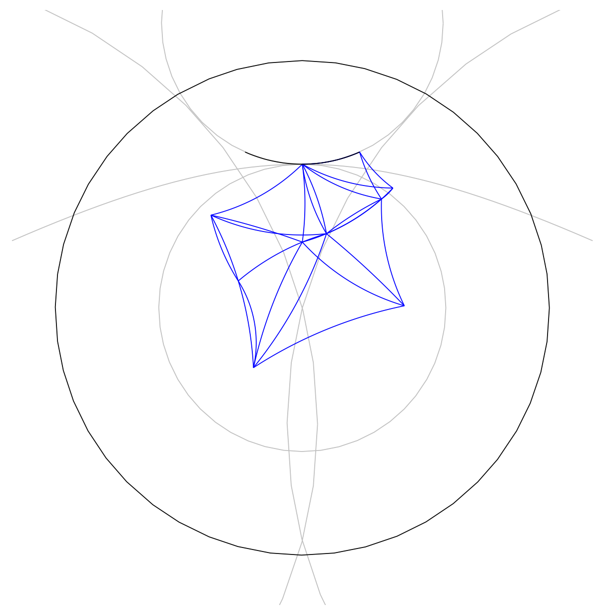

Figure A.16: Projection of the 1-skeleton of $P^2(\Delta)$ onto Σ_A

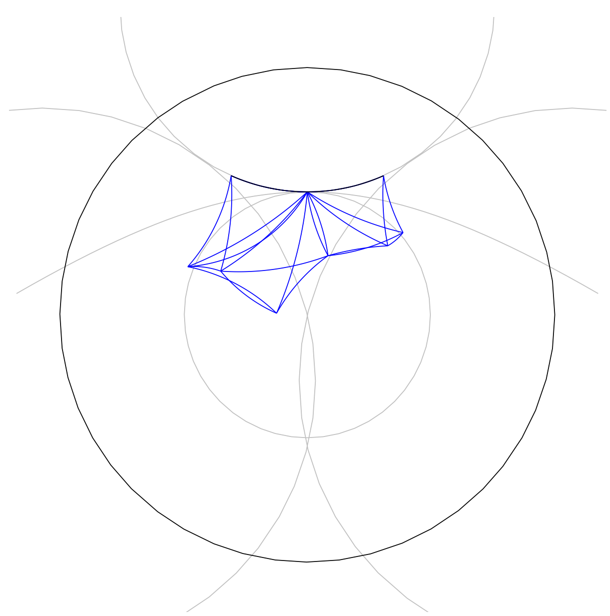

Figure A.17: Projection of the 1-skeleton of $P^2(\Delta)$ onto Σ_B

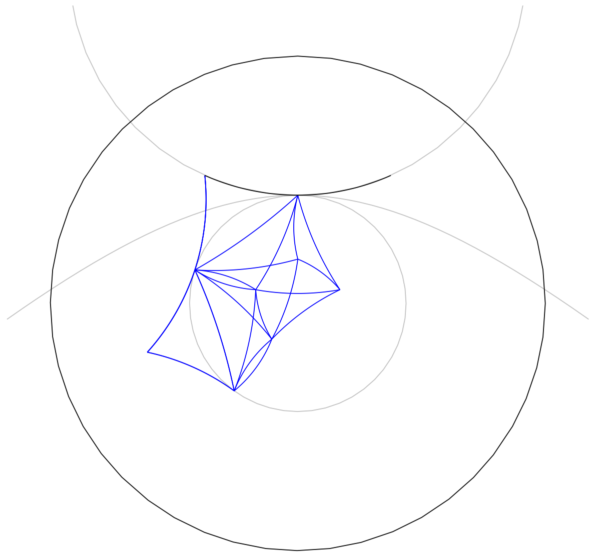

Figure A.18: Projection of the 1-skeleton of $P^2(\Delta)$ onto Σ_C

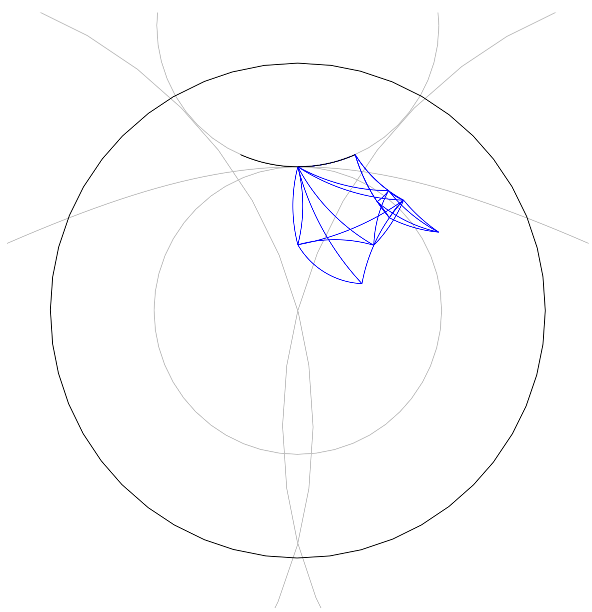

Figure A.19: Projection of the 1-skeleton of $P^3(\Delta)$ onto Σ_A

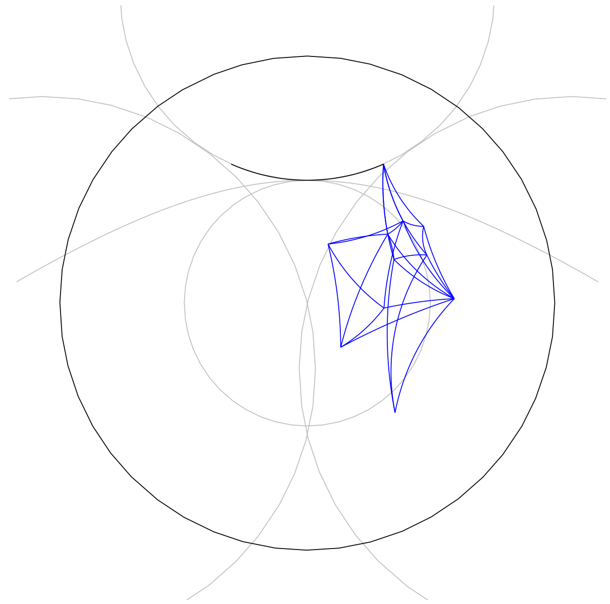

Figure A.20: Projection of the 1-skeleton of $P^3(\Delta)$ onto Σ_B

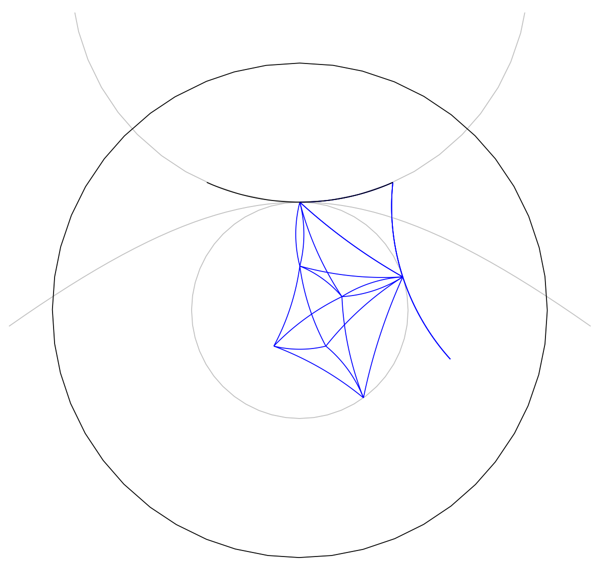

Figure A.21: Projection of the 1-skeleton of $P^3(\Delta)$ onto Σ_C

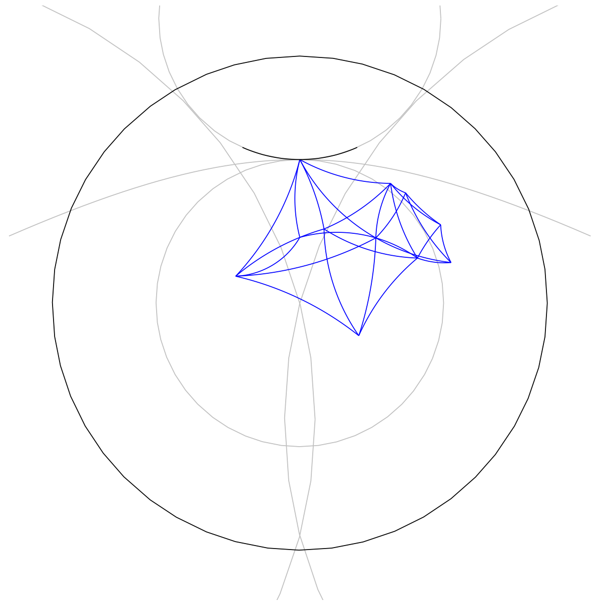

Figure A.22: Projection of the 1-skeleton of $P^5(\Delta)$ onto Σ_A

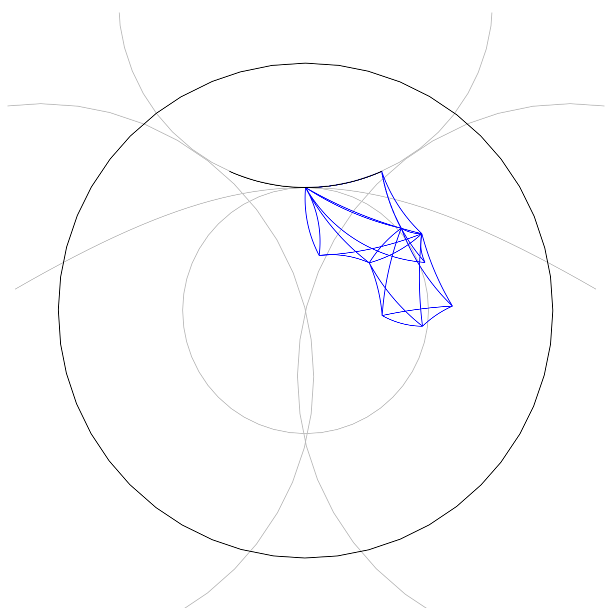

Figure A.23: Projection of the 1-skeleton of $P^5(\Delta)$ onto Σ_B

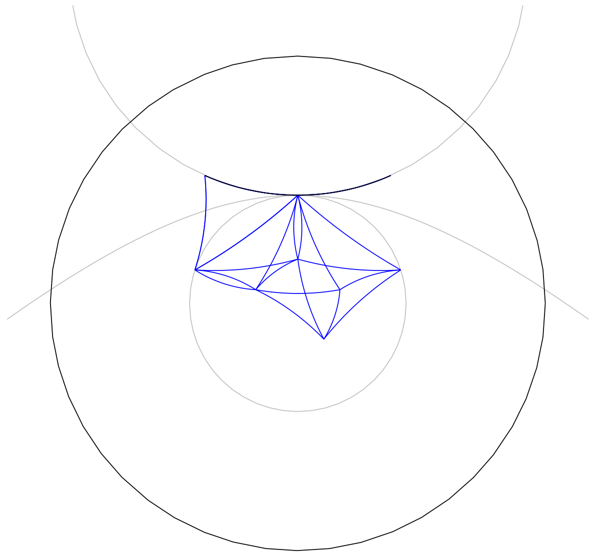

Figure A.24: Projection of the 1-skeleton of $P^5(\Delta)$ onto Σ_C

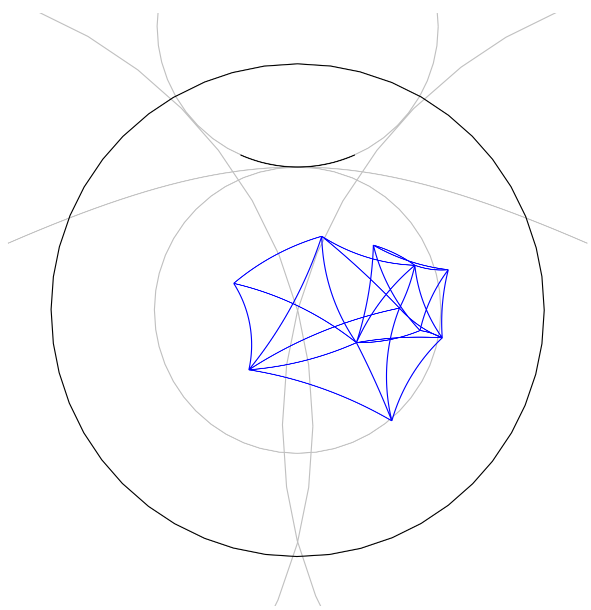

Figure A.25: Projection of the 1-skeleton of $P^7(\Delta)$ onto Σ_A

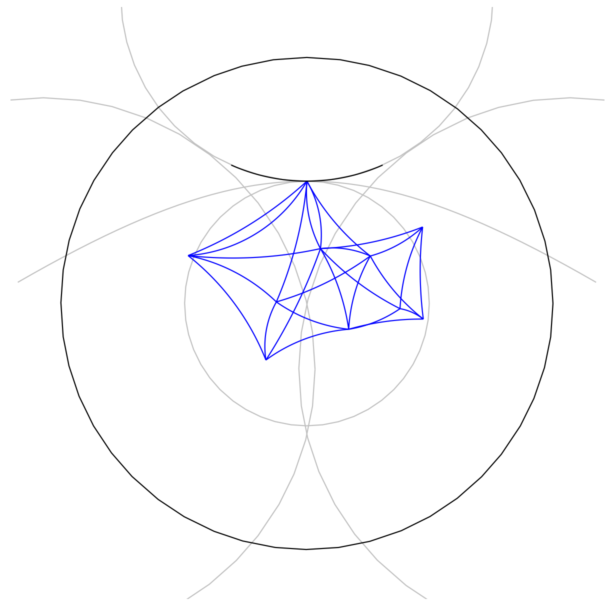

Figure A.26: Projection of the 1-skeleton of $P^7(\Delta)$ onto Σ_B

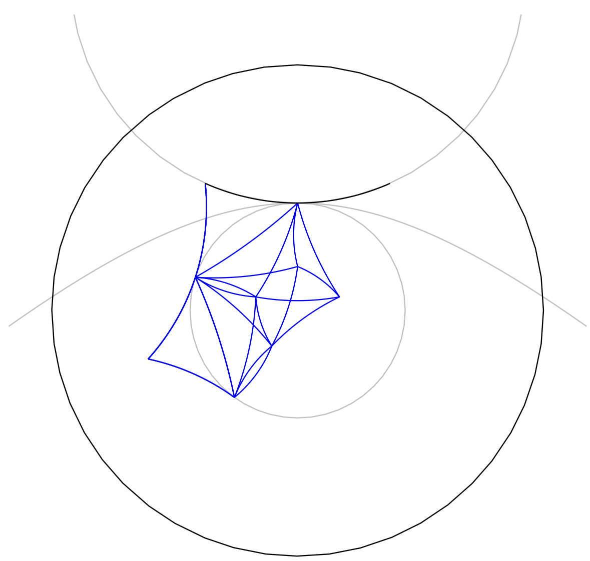

Figure A.27: Projection of the 1-skeleton of $P^7(\Delta)$ onto Σ_C

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